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THEORIES OF NUTATION AND POLAR MOTION II

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The report describes and interrelates various theories of nutation and polar motion based on an earth model consisting of an elastic mantle and a liquid core. In order to make basic features transparent and to facilitate interrelation and intercomparison, the liquid core is considered homogeneous and incompressible. A detailed treatment of the theory of Molodensky serves as a basis for describing recent advances by Shen and Mansinha, Smith, Wahr, and others; an elementary treatment of the variational approach by Jeffreys and Vicente and a numerical comparison conclude the report.

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FOREWORD

This report was prepared by Dr. Helmut Moritz, Professor, Technical University at Graz and Adjunct Professor, Department of Geodetic Science and Surveying of The Ohio State University, under Air Force Contract No. F19628-79-C-0075, The Ohio State University Research Foundation, Project No. 711715, Project Supervisor, Urho A. Uotila, Professor, Department of Geodetic Science and Surveying. The contract covering this research is administered by the Air Force Geophysics Laboratory, Hanscom Air Force Base, MA, with Mr. Bela Szabo, Contract Monitor.

CONTENTS

INTRODUCTION

This report is a continuation of the author's "Theories of Nutation and Polar Motion I", Report No. 309, Department of Geodetic Science, The Ohio State University, 1980. It considers theories based on earth models consisting of an elastic mantle and a liquid core.

The first theoretical treatment of such a model has been given already by Poincaré (1910), but only to the extent to show that a nearly diurnal free wobble (NDFW) exists. His treatment is sketchy and incomplete; his approach has not been followed by later authors. No quantitative results are given.

Only in 1949, Harold Jeffreys took up the problem again and used a variational method, although a somewhat simplified one, to get preliminary numerical values. A perfected version of the variational approach (Jeffreys and Vicente, 1957a,b) gave the first numerical results that are so detailed and accurate as to come close to modern standards, although the underlying earth models were rather simplified.

The variational approach of Jeffreys and Vicente has not been pursued since. Molodensky (1961) writes: "Jeffreys and Vicente applied a variational method which made it difficult to have a clear notion about the degree of approximation to the exact solution of the problem. I attempted to reproduce this theory and to make it more exact, but owing to numerous obscurities I had to give up this method."

Thus Molodensky (1961) developed a method of his own, based

on a spherical approximation of the equations of elasticity for the mantle (and, possibly, for a solid inner core) and on a hydrodynamical theory for the liquid core. He got numerical results which are excellent even by present standards; furthermore, this approach -- generalized, modified, simplified -- has been the starting point of practically all subsequent work.

Unfortunately, the classical papers by Jeffreys-Vicente and Molodensky are very difficult to read, because of "certain expository difficulties in both", as Smith (1977) remarks. On the other hand, they are important enough to deserve careful study.

The present report attempts an elementary introduction to the work of Jeffreys, Molodensky and subsequent developments. We shall try to interrelate various approaches by using a basic set of parameters (essentially Molodensky's) and relating to it other variables such as employed by Jeffreys. The various coordinate systems used by different authors will be seen to reduce to two basic systems, the nutation frame (Jeffreys, Smith, Wahr) and the body frame (Molodensky, Shen-Mansinha), which are easily related.

The most difficult part of (Molodensky, 1961) is the hydrodynamic theory of the core. In order to clarify the essential features and to supply motivations which are missing in Molodensky's presentation, we present a pedestrian's version using a simplified model of a homogeneous core. This model, although inadequate from a practical point of view, is best suited for a first approach; it permits an instructive comparison of the methods of Jeffreys and Molodensky with the classical theory of Poincaré (for a rigid mantle). We shall also show the relation of Molodensky's theory to modern treatments of the core in terms of toroidal oscillations (Smith, Shen and Mansinha, Wahr).

PART A

HYDRODYNAMICS OF A HOMOGENEOUS CORE

1. Hydrodynamic Equations

In an inertial frame XYZ, the motion of an ideal (non-viscous) fluid is described by the well-known equations

$$\ddot{\underline{X}} = \operatorname{grad} V - \rho^{-1} \operatorname{grad} p \; ; \qquad (1\text{-}1)$$

cf. (Lamb, 1932, pp. 3-4; Lanczos, 1970, p. 361). Here

$$\ddot{\underline{X}} = d^2\underline{X}/dt^2 \qquad (1\text{-}2)$$

denotes the second derivative of the position vector $\underline{X} = (X,Y,Z)$ with respect to time; it represents the acceleration of a moving particle, d/dt denoting a Lagrangian differentiation following the motion of the particle (as opposed to an Eulerian differentiation referred to a point of fixed coordinates). The vector $\operatorname{grad} V$ is the gradient of a potential V,

$$\operatorname{grad} V = (\partial V/\partial X,\ \partial V/\partial Y,\ \partial V/\partial Z)\ , \tag{1-3}$$

ρ is the density and p the pressure of the fluid.

In a moving frame xyz, rotating with an angular velocity vector $\underline{\omega}$, equations (1-1) are to be replaced by

$$\ddot{\underline{x}} + \dot{\underline{\omega}}\times\underline{x} + 2\underline{\omega}\times\dot{\underline{x}} + \underline{\omega}\times(\underline{\omega}\times\underline{x}) = \operatorname{grad} V - \rho^{-1}\operatorname{grad} p\ , \tag{1-4}$$

where on the left-hand side there occur additional accelerations, which may be considered apparent or inertial forces:

$$\begin{aligned} &\dot{\underline{\omega}}\times\underline{x} \quad \ldots\ldots\ldots\ldots\ldots\ \text{Euler force},\\ &2\underline{\omega}\times\dot{\underline{x}} \quad \ldots\ldots\ldots\ldots\ldots\ \text{Coriolis force},\\ &\underline{\omega}\times(\underline{\omega}\times\underline{x}) \quad \ldots\ldots\ldots\ \text{centrifugal force;} \end{aligned} \tag{1-5}$$

cf. (Lanczos, 1970, p. 101). The cross ($\times$) denotes the vector product as usual.

Let us take the z-axis coinciding with the rotation axis; then $\underline{\omega}$ has the form

$$\underline{\omega} = (0,0,\Omega)\ . \tag{1-6}$$

We furthermore assume constant rotation so that $\dot{\underline{\omega}} = 0$. Then

(1-4) takes the form

$$\begin{aligned}
\ddot{x} - 2\Omega\dot{y} - \Omega^2 x &= \frac{\partial V}{\partial x} - \rho^{-1}\frac{\partial p}{\partial x} \ , \\
\ddot{y} + 2\Omega\dot{x} - \Omega^2 y &= \frac{\partial V}{\partial y} - \rho^{-1}\frac{\partial p}{\partial y} \ , \qquad (1\text{-}7) \\
\ddot{z} \qquad\qquad\quad &= \frac{\partial V}{\partial z} - \rho^{-1}\frac{\partial p}{\partial z} \ .
\end{aligned}$$

The vector $(\Omega^2 x, \Omega^2 y, 0)$ is the gradient vector of the centrifugal potential

$$\phi = \frac{1}{2}\Omega^2(x^2+y^2) \ . \qquad (1\text{-}8)$$

We furthermore assume a homogeneous fluid with

$$\rho = \text{const.} \ , \qquad (1\text{-}9)$$

and we introduce the function

$$-\psi = V + \phi - \rho^{-1}p \qquad (1\text{-}10)$$

and the displacements

$$\begin{aligned}
u &= x - x_0 \ , \\
v &= y - y_0 \ , \qquad\qquad \underline{u} = \underline{x} - \underline{x}_0 \qquad (1\text{-}11) \\
w &= z - z_0 \ ,
\end{aligned}$$

of a fluid particle with respect to some fixed initial position $\underline{x}_0$ so that

$$\dot{\underline{x}} = \dot{\underline{u}} , \qquad \ddot{\underline{x}} = \ddot{\underline{u}} . \tag{1-12}$$

Then the system (1-7) reduces to

$$\begin{aligned} \ddot{u} - 2\Omega\dot{v} &= -\partial\psi/\partial x , \\ \ddot{v} + 2\Omega\dot{u} &= -\partial\psi/\partial y , \\ \ddot{w} \qquad &= -\partial\psi/\partial z , \end{aligned} \tag{1-13}$$

or in vector notation,

$$\ddot{\underline{u}} + 2\underline{\omega}\times\dot{\underline{u}} = -\operatorname{grad}\psi . \tag{1-14}$$

This form has been used by Hough (1895, p. 479) in his pioneering investigation of an earth model featuring a rigid mantle and a liquid core; cf. also (Moritz, 1980b, p. 100).

An important special solution of the system (1-13) is obtained by putting

$$\psi = B(xz\cos\sigma t + yz\sin\sigma t) \tag{1-15}$$

with constants B and σ . Then

$$u = \frac{B}{\sigma(\sigma+2\Omega)} z \cos \sigma t \; ,$$

$$v = \frac{B}{\sigma(\sigma+2\Omega)} z \sin \sigma t \; , \qquad (1\text{-}16)$$

$$\psi = \frac{B}{\sigma^2} (x \cos \sigma t + y \sin \sigma t)$$

satisfy (1-13), as we immediately verify by substitution.

Nutation frame and body frame. In the present report, we shall use two basic reference frames in which the earth's mantle is approximately at rest. The first is the body frame $x_1x_2x_3$ which, for a rigid mantle, is rigidly connected to it in such a way that the x_3 axis is the symmetry axis (figure axis) of the earth, supposed to be an ellipsoid of revolution. For an elastic mantle, x_3 represents the Liouville axis which corresponds to the figure axis of the undeformed earth; it is the same as the z-axis in (Moritz, 1980b, p. 49). This frame is appropriate for polar motion, which is motion around the x_3-axis.

In (Moritz, 1980b, pp. 76) we have introduced, in addition to the body frame $x_1x_2x_3$, an auxiliary system $x_1^0x_2^0x_3^0$ which is connected to the inertial system in a prescribed way: the x_3^0-axis has a fixed direction in inertial space, and the system $x_1^0x_2^0x_3^0$ rotates with constant angular velocity Ω around the x_3^0-axis. This system is appropriate for the treatment of nutation,

which is motion around the x_3^o axis (Moritz, 1980b, p. 92). Therefore, the auxiliary uniformly rotating system $x_1^o x_2^o x_3^o$ will be called <u>nutation frame</u>.

The body frame and the nutation frame differ only by a small rotation:

$$\underline{x} = (\underline{I} + \underline{\Theta})\underline{x}^o \tag{1-17}$$

where

$$\underline{\Theta} = \begin{bmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{bmatrix} \tag{1-18}$$

is a small skew-symmetric matrix.

In view of (1-6) it is clear that the equation of motion <u>in the nutation frame</u> has the form (1-14).

<u>Equation of motion in the body frame</u>. This equation has a different form. To derive it, we put

$$\underline{\omega} = \underline{\omega}_0 + \delta\underline{\omega} \ , \tag{1-19a}$$

$$\underline{\omega}_0 = (0,\ 0,\ \Omega) \ , \qquad \delta\underline{\omega} = \Omega(m_1,\ m_2,\ m_3) \ . \tag{1-19b}$$

We shall only deal with polar motion, which is the change of <u>direction</u> of the vector $\underline{\omega}$ described by m_1 and m_2, but not with irregularities in the angular velocity, which is the change of <u>magnitude</u> of the vector $\underline{\omega}$ described by m_3. In agreement with (Moritz, 1980b, pp. 19 and 85), we, therefore, put

$$m_3 = 0 \ . \tag{1-19c}$$

The use of the new system xyz means that we have to consider $\delta\underline{\omega}$ and add its effect to the left hand side of (1-7) which by (1-6) corresponds to $\underline{\omega} = \underline{\omega}_0$. By (1-5), we have an Euler force

$$\underline{f}_E = \delta\dot{\underline{\omega}}\times\underline{x} \ , \tag{1-20}$$

a change of Coriolis force, using (1-12),

$$\delta\underline{f}_{CO} = 2\ \delta\underline{\omega}\times\dot{\underline{x}} = 2\ \delta\underline{\omega}\times\dot{\underline{u}} \ , \tag{1-21}$$

and a change of centrifugal force,

$$\delta\underline{f}_C = \underline{\omega}\times(\underline{\omega}\times\underline{x}) - \underline{\omega}_0\times(\underline{\omega}_0\times\underline{x}) \ . \tag{1-22}$$

We shall consider $\delta\underline{\omega}$ and $\underline{u}$, as well as their derivatives, as quantities of first order and shall consistently neglect terms

of second and higher order. Then (1-21), being a product of first order terms, is of second order and will be neglected. Thus,

$$\delta \underline{f}_{CO} = 0 \ . \tag{1-23}$$

To the same approximation,

$$\delta \underline{f}_C = \delta\underline{\omega}\times(\underline{\omega}_0\times\underline{x}) + \underline{\omega}_0\times(\delta\underline{\omega}\times\underline{x}) \ , \tag{1-24}$$

which is found by substituting (1-19a) into (1-22) and neglecting the quadratic term $\delta\underline{\omega}\times(\delta\underline{\omega}\times\underline{x})$.

We work out the vector products in formulas (1-20) and (1-24) using

$$\delta\underline{\omega} = (\Omega m_1 , \Omega m_2 , 0) \tag{1-25}$$

by (1-19a,b). The result is

$$\underline{f}_E = \begin{bmatrix} \dot{\Omega} m_2 z \\ -\dot{\Omega} m_1 z \\ \dot{\Omega}(-m_2 x + m_1 y) \end{bmatrix} , \tag{1-26}$$

$$\delta \underline{f}_C = \begin{bmatrix} \Omega^2 m_1 z \\ \Omega^2 m_2 z \\ \Omega^2 (m_1 x + m_2 y) \end{bmatrix} . \tag{1-27}$$

(We shall indifferently use row or column notation for vectors, whichever is more convenient.)

We put

$$\begin{aligned} m_1 &= \varepsilon \cos\sigma t \ , \\ m_2 &= \varepsilon \sin\sigma t \ . \end{aligned} \tag{1-28}$$

Since m_1 and m_2 are the rectangular coordinates of the pole of rotation in the tangential plane at the unit sphere at its intersection with the z-axis, (1-28) corresponds to a circular polar motion with (angular) frequency σ and radius ε. The situation is comparable to Eulerian motion illustrated in (Moritz, 1980b, p. 11, Fig. 2.1), but ε and σ are general and different from α and σ_E.

Then (1-27) becomes

$$\delta \underline{f}_C = \begin{bmatrix} \Omega^2 \varepsilon \, z \cos\sigma t \\ \Omega^2 \varepsilon \, z \sin\sigma t \\ \Omega^2 \varepsilon \, (x \cos\sigma t + y \sin\sigma t) \end{bmatrix} \tag{1-29}$$

which can be expressed in the form

$$\delta \underline{f}_C = -\,\mathrm{grad}\,\phi \, , \tag{1-30}$$

where

$$\phi = -\Omega^2 \epsilon (xz \cos \sigma t + yz \sin \sigma t) \tag{1-31}$$

is an <u>incremental centrifugal potential</u>. The Euler force (1-26), on using (1-28), takes the form

$$\underline{f}_E = \begin{bmatrix} \Omega\sigma\epsilon z \cos \sigma t \\ \Omega\sigma\epsilon z \sin \sigma t \\ -\Omega\sigma\epsilon (x \cos \sigma t + y \sin \sigma t) \end{bmatrix} \tag{1-32}$$

which can be expressed as

$$\underline{f}_E = -\frac{\sigma}{\Omega}\,\mathrm{grad}\phi + 2\,\frac{\sigma}{\Omega}\,\frac{\partial \phi}{\partial z}\,\underline{e}_3 \, , \tag{1-33}$$

where $\underline{e}_3 = (0,0,1)$ is the unit vector of the z-axis. It is immediately verified that (1-33), with (1-31), gives (1-32).

Thus the quantity

$$\underline{f}_E + \delta\underline{f}_C = -(1+\frac{\sigma}{\Omega})\ \mathrm{grad}\phi + 2\,\frac{\sigma}{\Omega}\,\frac{\partial\phi}{\partial z}\,\underline{e}_3 \qquad (1\text{-}34)$$

must be added to the left hand side of (1-14), in which $\underline{\omega}$ is understood as $\underline{\omega}_0$. This gives

$$\ddot{\underline{u}} + 2\underline{\omega}_0\times\underline{u} - (1+\frac{\sigma}{\Omega})\ \mathrm{grad}\phi + 2\,\frac{\sigma}{\Omega}\,\frac{\partial\phi}{\partial z}\,\underline{e}_3 = -\mathrm{grad}\psi_0\ , \qquad (1\text{-}35)$$

where ψ of eq. (1-10) is now denoted by ψ_0 :

$$-\psi_0 = V + \Phi - \rho^{-1}p\ ; \qquad (1\text{-}36)$$

Φ is again defined by (1-8). Let us finally define a new function ψ by

$$-\psi = -\psi_0 + (1+\frac{\sigma}{\Omega})\phi = V + \Phi - \rho^{-1}p + (1+\frac{\sigma}{\Omega})\phi\ . \qquad (1\text{-}37)$$

Then (1-35) takes the final form

$$\ddot{\underline{u}} + 2\underline{\omega}_0\times\underline{u} = -\mathrm{grad}\psi - 2\,\frac{\sigma}{\Omega}\,\frac{\partial\phi}{\partial z}\,\underline{e}_3\ , \qquad (1\text{-}38)$$

which will be used in the following section.

2. Method of Molodensky

Molodensky's method is not restricted to fluids of constant density; for a homogeneous core, however, it becomes essentially simpler. Thus we shall here, and throughout the present report, presuppose a core consisting of an ideal homogeneous fluid. This will provide an easy introduction to an otherwise very difficult theory and thus facilitate the reading of presentations such as (Molodensky, 1961; Jobert, 1964; Melchior, 1978, sec. 6.2).

The system of differential equations of fluid motion (1-38) may be written

$$\begin{aligned} \ddot{u} - 2\Omega\dot{v} &= -\frac{\partial\psi}{\partial x} , \\ \ddot{v} + 2\Omega\dot{u} &= -\frac{\partial\psi}{\partial y} , \\ \ddot{w} &= -\frac{\partial\psi}{\partial z} - 2\,\frac{\sigma}{\Omega}\,\frac{\partial\varphi}{\partial z} , \end{aligned} \tag{2-1}$$

where

$$-\psi = V + \phi - \rho^{-1}p + \left(1 + \frac{\sigma}{\Omega}\right)\varphi \quad , \tag{2-2}$$

$$\phi = \frac{1}{2}\Omega^2(x^2 + y^2) \; , \tag{2-3}$$

$$\varphi = -\Omega^2\varepsilon(xz\cos\sigma t + yz\sin\sigma t) \; , \tag{2-4}$$

V is the potential of gravitational attraction acting on the fluid module under consideration, and p is the pressure; the density ρ is constant.

The coordinate system xyz is such that the mantle, on the average, is at rest with respect to it, and the z-axis has a constant direction with regard to inertial space. The unit vector of the instantaneous rotation axis has the x and y coordinates

$$m_1 = \varepsilon\cos\sigma t \; , \qquad m_2 = \varepsilon\sin\sigma t \; . \qquad (2\text{-}5)$$

If the rotation axis coincides with the z-axis, for $\varepsilon = 0$, then eqs. (2-1) reduce to (1-13).

Let us now consider harmonic oscillations of the form

$$\underline{u}(x,y,z,t) = \underline{u}_1(x,y,z)\cos\sigma t + \underline{u}_2(x,y,z)\sin\sigma t \; , \qquad (2\text{-}6)$$

depending on time with (angular) frequency σ. Then

$$\ddot{\underline{u}} = -\sigma^2\underline{u} \; , \qquad (2\text{-}7)$$

that is,

$$\ddot{u} = -\sigma^2 u \; , \qquad \ddot{v} = -\sigma^2 v \; , \qquad \ddot{w} = -\sigma^2 w \; , \qquad (2\text{-}8)$$

as we directly see by substitution. Thus (2-1) becomes

$$\begin{aligned} \sigma^2 u + 2\Omega\dot{v} &= \psi_x \ , \\ \sigma^2 v - 2\Omega\dot{u} &= \psi_y \ , \\ \sigma^2 w &= \psi_z + 2\,\frac{\sigma}{\Omega}\,\phi_z \ . \end{aligned} \tag{2-9}$$

Differentiation with respect to time, using (2-8), gives

$$\begin{aligned} \sigma^2 \dot{u} - 2\Omega\sigma^2 v &= \dot{\psi}_x \ , \\ \sigma^2 \dot{v} + 2\Omega\sigma^2 u &= \dot{\psi}_y \ , \end{aligned} \tag{2-10}$$

where

$$\psi_x = \frac{\partial\psi}{\partial x} \ , \qquad \dot{\psi}_x = \frac{\partial^2\psi}{\partial x\partial t} \ , \quad \text{etc.} \tag{2-11}$$

We solve (2-10) for $\dot{u}$ and $\dot{v}$ and substitute in (2-9). This gives us

$$\begin{aligned} (\sigma^2 - 4\Omega^2)u &= \psi_x - 2\,\frac{\Omega}{\sigma^2}\,\dot{\psi}_y \ , \\ (\sigma^2 - 4\Omega^2)v &= \psi_y + 2\,\frac{\Omega}{\sigma^2}\,\dot{\psi}_x \ , \\ \sigma^2 w &= \psi_z + 2\,\frac{\sigma}{\Omega}\,\phi_z \ , \end{aligned} \tag{2-12}$$

which solves (2-1) for u, v, w .

The <u>volume dilatation</u> is defined by

$$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \; ; \qquad (2\text{-}13)$$

it characterizes the change of a volume element of the fluid during motion. A <u>homogeneous</u> fluid for which

$$\rho = \text{const.} \qquad (2\text{-}14)$$

in space and time, is necessarily <u>incompressible</u>. This means

$$\theta = 0 \; . \qquad (2\text{-}15)$$

We take u, v, w from (2-12), substitute into (2-13) and put the result equal to zero:

$$\frac{1}{\sigma^2 - 4\Omega^2} \left(\psi_{xx} - 2\,\frac{\Omega}{\sigma^2}\,\dot{\psi}_{xy} + \psi_{yy} + 2\,\frac{\Omega}{\sigma^2}\,\dot{\psi}_{xy}\right) +$$

$$+ \frac{1}{\sigma^2}\left(\psi_{zz} + 2\,\frac{\sigma}{\Omega}\,\phi_{zz}\right) = 0 \; .$$

Since $\phi_{zz} = 0$ by (2-4), we are left with <u>Poincaré's equation</u>

$$\psi_{xx} + \psi_{yy} + \left(1 - 4\,\frac{\Omega^2}{\sigma^2}\right)\psi_{zz} = 0 \qquad (2\text{-}16)$$

(Poincaré, 1885; Hough, 1895; Jeffreys, 1949; Molodensky, 1961).

This equation is similar to Laplace's equation, to which it reduces for $\Omega = 0$. Laplace's equation is known to have 5 independent solutions of second degree:

$$
\begin{aligned}
&z^2 - (x^2 + y^2)/2 \,, \\
&xz, \quad yz, \\
&(x^2 - y^2)/2, \quad xy \;;
\end{aligned}
\tag{2-17}
$$

cf. (Heiskanen and Moritz, 1967, p. 61), constant factors do not matter. Similarly, Poincaré's equation has the solutions

$$
\begin{aligned}
&z^2 - \frac{1}{2}(1 - 4\Omega^2/\sigma^2)(x^2 + y^2) \,, \\
&xz, \quad yz, \\
&\frac{1}{2}(x^2 - y^2), \quad xy \,,
\end{aligned}
\tag{2-18}
$$

as we see by substituting into (2-16). To get an appropriate time dependence, these solutions must be multiplied by $\cos\sigma t$ or $\sin\sigma t$.

We shall take the solution

$$\psi = B(xz\cos\sigma t + yz\sin\sigma t) \,, \tag{2-19}$$

which is a linear combination of the solutions xz and yz.

Using (2-4) and (2-19) in (2-12) we get

$$(\sigma^2 - 4\Omega^2)u = B(1 - 2\Omega/\sigma)z\cos\sigma t \quad ,$$

$$(\sigma^2 - 4\Omega^2)v = B(1 - 2\Omega/\sigma)z\sin\sigma t \quad ,$$

$$\sigma^2 w = (B - 2\sigma\Omega\epsilon)(x\cos\sigma t + y\sin\sigma t) \quad .$$

By means of the substitution

$$\beta = \frac{B}{\sigma(\sigma + 2\Omega)} \tag{2-20}$$

this becomes

$$\begin{aligned} u &= \beta z\cos\sigma t \quad , \\ v &= \beta z\sin\sigma t \quad , \\ w &= \left(\frac{\sigma + 2\Omega}{\sigma}\beta - \frac{2\Omega}{\sigma}\epsilon\right)(x\cos\sigma t + y\sin\sigma t) \quad . \end{aligned} \tag{2-21}$$

For $\epsilon = 0$, this reduces to (1-16), as it should.

<u>Actual state and reference state</u>. We shall introduce a "reference state" from which we count deformations. It corresponds to the absence of external forces or deformations ("undeformed state"). We shall denote quantities belonging to the reference state by the subscript 0 .

Thus we may split up the total potential V acting on a point of the deformed liquid body:

$$V = V_0 + V_e + V_1 \quad . \tag{2-22}$$

V_0 is the attractional potential of the undeformed body, V_e is the potential of external forces (tidal potential), and V_1 is the change of attractional potential of the body due to the change of its shape caused by tidal deformations.

The pressure p is split up similarly :

$$p = p_0 + p_1 , \tag{2-23}$$

where p is the pressure in the deformed body, and p_0 is the pressure in the undeformed body; p_1 is a small correction. The constant density ρ is assumed the same in the reference state as in the actual state.

Introduce the gravity potential W_0 in the undeformed body as the sum of gravitational potential V_0 and centrifugal potential Φ :

$$W_0 = V_0 + \Phi . \tag{2-24}$$

Let us further assume that the reference state corresponds to hydrostatic equilibrium, in which the particles are at rest. Rotation is about the z-axis, so that ε_0 is zero. Then

$$\dot{\underline{u}}_0 = 0$$

since there is no particle motion,

$$\phi_0 = 0$$

by (2-4) since $\varepsilon_0 = 0$, and (1-38) gives

$$0 = -\mathrm{grad}\,\psi_0 \tag{2-25}$$

where

$$-\psi_0 = V_0 + \Phi - \rho^{-1}p_0 = W_0 - \rho^{-1}p_0 \tag{2-26}$$

by (2-2) for the reference state (note that the present ψ_0 is not quite the same as (1-36)).

The substitution of (2-26) into (2-25) gives

$$\mathrm{grad}\, W_0 = \frac{1}{\rho}\,\mathrm{grad}\, p_0 \quad , \tag{2-27}$$

which is the well-known <u>condition of hydrostatic equilibrium</u>: cf. (Jeffreys, 1962, sec. 4.03). The physical meaning of this equation can be seen by writing it as

$$d\,W_0 = \frac{1}{\rho}\,d\,p_0 \quad . \tag{2-28}$$

Thus $d\,p_0 = 0$ implies $dW = 0$ and vice versa : <u>the surfaces of constant potential coincide with the surfaces of constant pressure</u>.

Consider now such a surface of constant reference potential Σ_0 , which is deformed into a surface Σ (Fig. 2.1). A point B of the actual state is separated from the corresponding reference point by the deformation vector $\underline{u}$.

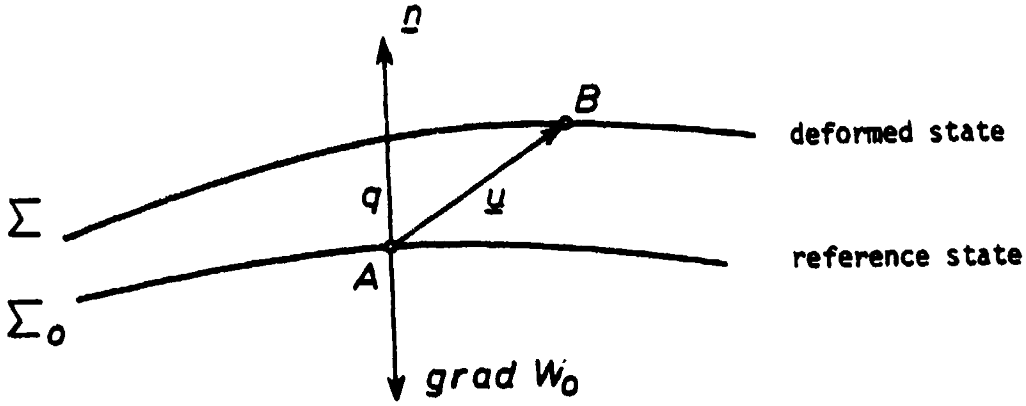

Fig. 2.1. Deformed state and reference state.

Consider now any quantity f of the actual state, and its corresponding quantity f_0 for the reference state. Differences of the form

$$f_A - f_{0_A} \qquad \text{or} \qquad f_B - f_{0_B} \tag{2-29}$$

in which f and f_0 refer to the same point, are called Eulerian differences, and differences of the form

$$f_B - f_{0_A} \ , \tag{2-30}$$

in which f refers to the "actual point" B and f_0 to its "reference point" A , are called Lagrangian differences. This is an established terminology in continuum mechanics.

To the geodesist, an analogy to the gravity field may be helpful. The actual gravity field is split up into a normal field and an anomalous field (Heiskanen and Moritz, 1967, sec. 2-13). The surface Σ may be compared to the geoid, and Σ_0 to the ellipsoid.

The anomalous potential T, for which actual potential W and normal potential U refer to the same point. is an "Eulerian" quantity, whereas the gravity anomaly Δg, for which actual gravity refers to a geodial point P and normal gravity γ to the corresponding ellipsoidal point Q, is a "Lagrangian" quantity.

We shall now make the definition of V_1 in (2-22) more precise by understanding it as an Eulerian quantity : V and the reference potential V_0 refer to the same point :

$$V_A = V_{0A} + V_{eA} + V_{1A} \quad . \tag{2-31}$$

On the other hand, the anomalous pressure p_1 in (2-23) is defined to be a Lagrangian increment :

$$p_1 = p_B - p_{0A} \cdot \tag{2-32}$$

Let us denote by η the difference

$$\eta = W_B - W_A \; .$$

By Taylor's theorem it is, to the linear approximation we are consistently using,

$$\eta = \underline{u} \cdot \text{grad}W \doteq \underline{u} \cdot \text{grad}\; W_0 \; . \tag{2-33}$$

Then

$$p_{0B} = p_{0A} + \rho\eta \ .$$

Finally,

$$p_1 = p_B - p_{0A} = p_B - p_{0B} + \rho\eta \ ,$$

so that

$$p_1 = \delta p + \rho\eta \tag{2-34}$$

links the Lagrangian increment p_1 to the Eulerian increment

$$\delta p = p - p_0 \ , \tag{2-35}$$

p and p_0 referring to the same point B . (The geodetic reader will note the analogy between p_1 and the gravity anomaly Δg , and between δp and the gravity disturbance δg !)

This simple but important relations enable us to transform expression (2-2) for our basic function ψ :

$$\begin{aligned} -\psi &= V + \Phi - \rho^{-1}p + (1 + \frac{\sigma}{\Omega}\phi) \\ &= V_0 + V_e + V_1 + \Phi - \rho^{-1}p_0 - \rho^{-1}\delta p + (1 + \frac{\sigma}{\Omega}\phi) \\ &= W_0 + V_e + V_1 + (1 + \frac{\sigma}{\Omega}\phi) - \frac{p_0}{\rho} - \frac{p_1}{\rho} + \eta \ . \end{aligned} \tag{2-36}$$

Here we have used (2-24), (2-34), and (2-35). The integration of (2-28) gives

$$W_0 = \frac{p_0}{\rho} \ , \tag{2-37}$$

apart from an integration constant which we can put equal to zero. The use of (2-37) in (2-36) gives finally

$$-\psi = V + n - \frac{p_1}{\rho} \ , \qquad (2\text{-}38)$$

if with Molodensky we put

$$V = V_e + V_1 + \left(1 + \frac{\sigma}{\Omega}\phi\right) \qquad (2\text{-}39)$$

(this is not the same V as in (2-22) above!).

External and induced potential. We assume an external (tidal) potential of the form

$$V_e(x,y,z) = \kappa(xz\cos\sigma t + yz\sin\sigma t) \ , \qquad (2\text{-}40)$$

κ being a constant. This corresponds to a tesseral tidal potential of degree 2 and order 1 ; cf. (Moritz, 1980b, p. 30). In fact, transforming to spherical coordinates r, θ, λ by

$$\begin{aligned} x &= r\sin\theta\cos\lambda \ , \\ y &= r\sin\theta\sin\lambda \ , \\ z &= r\cos\theta \ , \end{aligned} \qquad (2\text{-}41)$$

we get

$$\begin{aligned} V_e(r,\theta,\lambda) &= \kappa r^2\sin\theta\cos\theta(\cos\sigma t\cos\lambda + \sin\sigma t\sin\lambda) \\ &= \kappa r^2\sin\theta\cos\theta\cos(\sigma t - \lambda) \ . \end{aligned}$$

With the abbreviation

$$S_1(\theta,\lambda,t) = \sin\theta\cos\theta\cos(\sigma t - \lambda) \tag{2-42}$$

this becomes

$$V_e = \kappa r^2 S_1 \ . \tag{2-43}$$

We readily see from (2-4) that ϕ is also proportional to $r^2 S_1$. Poisson's equation (Heiskanen and Moritz, 1967, p. 5) gives

$$\Delta V_0 = -4\pi G\rho \ ,$$

where Δ denotes Laplace's operator and G the gravitational constant. After deformation, the density ρ remains the same. Thus, also

$$\Delta(V_0 + V_1) = -4\pi G\rho \ ,$$

whence on subtraction

$$\Delta V_1 = 0 \ , \tag{2-44}$$

which is Laplace's equation for harmonic functions. Hence V_1 is harmonic and may be taken to be proportional to

$$r^2 S_1 = xz\cos\sigma t + yz\sin\sigma t \ , \tag{2-45}$$

which clearly is harmonic.

Hence V as given by (2-39) will be proportional to (2-45). The expression (2-19) for $\dot{\psi}$ also has this form, so that we may put with Molodensky

$$V + \psi = A(xz\cos\sigma t + yz\sin\sigma t) . \tag{2-46}$$

It will be seen that

$$\eta = -C(xz\cos\sigma t + yz\sin\sigma t) \tag{2-47}$$

is consistent with these assumptions. Inserting (2-46) and (2-47) into (2-38) we find that the anomalous pressure

$$p_1 = \rho(A - C)(xz\cos\sigma t + yz\sin\sigma t) , \tag{2-48}$$

is also proportional to (2-45).

A boundary condition. The normal displacement q at the core-mantle boundary, which is assumed to be an equipotential ellipsoid, may be obtained in two ways. The first uses the fact that the boundary surface is an equipotential surface, the second, that it is an ellipsoid.

First, let Σ_0 and Σ in Fig. 2.1 denote this boundary surface in its undeformed and deformed position. Then the unit normal vector $\underline{n}$ is given by

$$\underline{n} = -\frac{\operatorname{grad} W_0}{|\operatorname{grad} W_0|} = -g^{-1}\operatorname{grad} W_0 , \tag{2-49}$$

if

$$g = |\text{grad}\, W_0|$$

denotes gravity (it is well known that the gravity vector $\text{grad}\, W_0$ is normal to the equipotential surface Σ_0). Thus, the dot denoting the inner product,

$$q = \underline{u} \cdot \underline{n} = -g^{-1}\underline{u} \cdot \text{grad}\, W_0 \quad , \tag{2-50}$$

and by (2-33),

$$q = -g^{-1}n \quad . \tag{2-51}$$

On the other hand, the boundary of the core is an ellipsoid of revolution

$$F(x,y,z) = \frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} - 1 = 0 \quad , \tag{2-52}$$

a and c denoting its semi-axes. The normal vector is from geometry given by

$$\underline{n} = \frac{\text{grad}\, F}{|\text{grad}\, F|} \quad . \tag{2-53}$$

With

$$e^2 = \frac{a^2 - c^2}{a^2} \quad ,$$

e denoting the excentricity, we have

$$\text{grad}\, F = \frac{2}{c^2} \begin{bmatrix} (1 - e^2)x \\ (1 - e^2)y \\ z \end{bmatrix} \quad . \tag{2-54}$$

To a sufficient approximation,

$$|\text{grad}\, F|^2 \doteq \frac{4}{c^4}(x^2 + y^2 + z^2) \doteq \frac{4}{c^2} \doteq \frac{4}{b^2} \tag{2-55}$$

since the ellipsoid is very nearly a sphere of radius $b = 3470$ km. Hence,

$$\underline{n} = \frac{1}{b} \begin{bmatrix} (1 - e^2)x \\ (1 - e^2)y \\ z \end{bmatrix} \quad . \tag{2-56}$$

Thus,

$$q = \underline{u} \cdot \underline{n}$$

gives

$$bq = (1 - e^2)(xu + yv) + zw \ , \tag{2-57}$$

and using (2-21) we find

$$bq = (1 - e^2)\beta(xz\cos\sigma t + yz\sin\sigma t) +$$

$$+ \left(\frac{\sigma + 2\Omega}{\sigma}\beta - \frac{2\Omega}{\sigma}\epsilon\right)(xz\cos\sigma t + yz\sin\sigma t) .$$

Putting

$$\nu = 2\left(\frac{\sigma + \Omega}{\sigma}\beta - \frac{\Omega}{\sigma}\epsilon\right) \tag{2-58}$$

this becomes

$$q = b^{-1}(\nu - e^2\beta)(xz\cos\sigma t + yz\sin\sigma t) . \tag{2-59}$$

For the principal lunisolar tides, $\sigma \doteq -\Omega$ (Moritz, 1980b, pp. 121-122), so that ν and $\nu - e^2\beta$ are small even if β is large, which can be the case as we shall see in sec.10. This shows that the approximation (2-55) is in fact justified.

The comparison between (2-51) and (2-59) gives

$$\eta = -gb^{-1}(\nu - e^2\beta)(xz\cos\sigma t + yz\sin\sigma t) . \tag{2-60}$$

The gravity g at a point of radius vector r at the interior of a homogeneous sphere is well known to be

$$g = \frac{4\pi G}{3}\rho r \tag{2-61}$$

(Kellogg, 1929, p. 19). Putting $r = b$ and substituting into (2-60) we get

$$\eta = -\frac{4\pi G}{3}\rho(\nu - e^2\beta)(xz\cos\sigma t + yz\sin\sigma t) . \qquad (2\text{-}62)$$

The comparison with (2-47) gives

$$C = \frac{4\pi G}{3}\rho(\nu - e^2\beta) . \qquad (2\text{-}63)$$

If we introduce a new constant γ by

$$C = -\frac{4\pi G}{3}\rho e^2\gamma , \qquad (2\text{-}64)$$

then (2-63) gives

$$(\gamma - \beta)e^2 + \nu = 0 . \qquad (2\text{-}65)$$

which can be considered a boundary condition expressing γ in terms of β and ϵ in view of (2-58).

<u>The constants A and α</u>. By (2-46) we have

$$V + \psi = A(xz\cos\sigma t + yz\sin\sigma t) .$$

Instead of A, let us introduce another constant α, which is related to A in the same way (2-64) as C is to γ:

$$A = -\frac{4\pi G}{3}\rho e^2\alpha . \qquad (2\text{-}66)$$

This parameter α was used by Molodensky (1961) who has put

$$V + \psi = \alpha\Phi \, , \qquad \Phi = K(r) r^{-2} (xz \cos\sigma t + yz \sin\sigma t) \; . \tag{2-67}$$

The function $K(r)$ satisfies a differential equation which for a homogeneous core becomes

$$K'' + \frac{2}{r} K' - \frac{6}{r^2} K = 0 \quad . \tag{2-68}$$

A solution is

$$K(r) = D r^2 \; . \tag{2-69}$$

Putting the constant $D = A\alpha^{-1}$, we get (2-46).

3. Toroidal and Spheroidal Modes

The modern treatment of core motion is in terms of toroidal and spheroidal oscillations (Shen and Mansinha, 1976; Smith, 1977; Wahr, 1979). We shall relate Molodensky's solution to this approach.

Toroidal and spheroidal modes are well known from the theory of free oscillations of the earth; cf. (Kaula, 1968, sec. 2.2). They were originally considered for an elastic body but can be applied to a liquid body as well, since the latter may be regarded as a limiting state of an elastic body; cf. sec. 5.

The displacement vector $\underline{u}$ is expressed as

$$\underline{u} = \sum_{n}\sum_{m} (\underline{S}_n^m + \underline{T}_n^m) \ , \tag{3-1}$$

$\underline{S}_n^m$ and $\underline{T}_n^m$ denoting, respectively, spheroidal and toroidal modes of degree and n and order m .

In spherical coordinates r, θ, λ, these vectors $\underline{S}_n^m$ and $\underline{T}_n^m$ have the components

$$\begin{aligned}
(\underline{S}_n^m)_r &= H_n^m(r) P_{nm}(\cos\theta)\cos(\sigma t - m\lambda) \ , \\
(\underline{S}_n^m)_\theta &= L_n^m(r) \frac{\partial}{\partial\theta} P_{nm}(\cos\theta)\cos(\sigma t - m\lambda) \ , \\
(\underline{S}_n^m)_\lambda &= mL_n^m(r) \frac{1}{\sin\theta} P_{nm}(\cos\theta)\sin(\sigma t - m\lambda) \ ;
\end{aligned} \tag{3-2}$$

$$(\underline{T}_n^m)_r = 0 ,$$
$$(\underline{T}_n^m)_\theta = -mT_n^m(r) \frac{1}{\sin\theta} P_{nm}(\cos\theta)\cos(\sigma t - m\lambda) , \qquad (3\text{-}3)$$
$$(\underline{T}_n^m)_\lambda = -T_n^m(r) \frac{\partial}{\partial\theta} P_{nm}(\cos\theta)\sin(\sigma t - m\lambda) .$$

We have followed the real notation of Shen and Mansinha (1976); other authors use a complex notation with

$$e^{i\sigma t} = \cos\sigma t + i \sin\sigma t \quad . \qquad (3\text{-}4)$$

The quantities $H_n^m, L_n^m,$ and T_n^m are functions of the radius vector r , and $P_{nm}(\cos\theta)$ are the usual Legendre functions; cf. Heiskanen and Moritz, 1967 sec. 1-11).

We consider only $\underline{S}_2^1$ and $\underline{T}_1^1$. With

$$P_{11}(\cos\theta) = \sin\theta , \qquad P_{21}(\cos\theta) = 3\sin\theta\cos\theta$$

we get from (3-2) and (3-3)

$$(\underline{T}_1^1)_r = 0 ,$$
$$(\underline{T}_1^1)_\theta = -T\cos(\sigma t - \lambda) , \qquad (3\text{-}5)$$
$$(\underline{T}_1^1)_\lambda = -T\cos\theta\sin(\sigma t - \lambda) ;$$

$$(\underline{S}_2^1)_r = 3H\cos\theta\sin\theta\cos(\sigma t - \lambda) ,$$
$$(3\text{-}6)$$

$$\left(\underline{S}_2^1\right)_\theta = 3L(\cos^2\theta - \sin^2\theta)\cos(\sigma t - \lambda) \quad , \tag{3-6}$$
$$\left(\underline{S}_2^1\right)_\lambda = 3L\cos\theta\sin(\sigma t - \lambda) \quad ,$$

where we have abbreviated

$$\begin{aligned} H_2^1(r) &= H(r) = H \ , \\ L_2^1(r) &= L(r) = L \ , \\ T_1^1(r) &= T(r) = T \ . \end{aligned} \tag{3-7}$$

The components of the vector $\underline{u}$ in spherical coodinates are transformed into its components u_1, u_2, u_3 in rectangular coordinates x,y,z by a rotation :

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\lambda & \cos\theta\cos\lambda & -\sin\lambda \\ \sin\theta\sin\lambda & \cos\theta\sin\lambda & \cos\lambda \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} u_r \\ u_\theta \\ u_\lambda \end{bmatrix} \quad ; \tag{3-8}$$

cf. (Heiskanen and Moritz, 1967), p. 206, equs. (5-54a,b,c), putting $\theta = 90^\circ - \phi$.

If we retain only toroidal oscillations (1,1) and spheroidal oscillations (2,1), then (3-1) reduces to

$$\underline{u} = \underline{T}_1^1 + \underline{S}_2^1 \tag{3-9}$$

and both vectors on the right-hand side are transformed by (3-8), so that we get rectangular components from spherical ones.

By a straightforward though somewhat laborious computation, using (2-41), we obtain

$$
\begin{aligned}
(\underline{T}_1^1)_1 &= -Tr^{-1} z \cos\sigma t \ , \\
(\underline{T}_1^1)_2 &= -Tr^{-1} z \sin\sigma t \ , \qquad (3\text{-}10) \\
(\underline{T}_1^1)_3 &= Tr^{-1}(x \cos\sigma t + y \sin\sigma t) \ ;
\end{aligned}
$$

$$
\begin{aligned}
(\underline{S}_2^1)_1 &= 3(H-2L)r^{-3}xz(x\cos\sigma t + y\sin\sigma t) + 3Lr^{-1}z\cos\sigma t \ , \\
(\underline{S}_2^1)_2 &= 3(H-2L)r^{-3}yz(x\cos\sigma t + y\sin\sigma t) + 3Lr^{-1}z\sin\sigma t \ , \\
& \qquad (3\text{-}11) \\
(\underline{S}_2^1)_3 &= [3(H-2L)r^{-3}z^2 + 3Lr^{-1}](x\cos\sigma t + y\sin\sigma t) \ .
\end{aligned}
$$

The solution (2-21) may be simplified by means of (2-58) :

$$
\begin{aligned}
u &= \beta z \cos\sigma t \ , \\
v &= \beta z \sin\sigma t \ , \qquad (3\text{-}12) \\
w &= (\nu - \beta)(x\cos\sigma t + y\sin\sigma t) \ .
\end{aligned}
$$

This may be written as follows:

$$\underline{u} = \underline{u}' + \underline{u}'' \ ; \qquad (3\text{-}13)$$

$$
\begin{aligned}
u' &= (\beta - \tfrac{1}{2}\nu) z \cos\sigma t \ , \\
v' &= (\beta - \tfrac{1}{2}\nu) z \sin\sigma t \ , \qquad (3\text{-}14) \\
w' &= -(\beta - \tfrac{1}{2}\nu)(x\cos\sigma t + y\sin\sigma t) \ ;
\end{aligned}
$$

$$u'' = \frac{1}{2}\,\nu z \cos\sigma t \quad ,$$
$$v'' = \frac{1}{2}\,\nu z \sin\sigma t \quad , \qquad (3\text{-}15)$$
$$w'' = \frac{1}{2}\,\nu (x \cos\sigma t + y \sin\sigma t) \quad .$$

The comparison with (3-10) and (3-11) shows that

$$\underline{u}' = \underline{T}_1^1 \quad , \qquad \underline{u}'' = \underline{S}_2^1 \qquad (3\text{-}16)$$

with $H = 2L$ and

$$H = \frac{1}{3}\,\nu r \quad ,$$
$$L = \frac{1}{6}\,\nu r \quad , \qquad (3\text{-}17)$$
$$T = -(\beta - \frac{1}{2}\,\nu) r \quad ,$$

where, by (2-58)

$$\nu = 2\left(\frac{\sigma + \Omega}{\sigma}\,\beta - \frac{\Omega}{\sigma}\,\varepsilon\right) . \qquad (3\text{-}18)$$

This shows that Molodensky's basic solution (2-21) may be interpreted as the resultant of a toroidal and a spheroidal oscillation, the functions (3-7) being simply proportional to r by (3-17).

Another insight is provided by considering the <u>matrix of deformation gradients</u>

$$\underline{U} = \left[\frac{\partial u_i}{\partial x_j}\right] = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix} . \qquad (3\text{-}19)$$

Here we have put

$$x_1 = x , \qquad x_2 = y , \qquad x_3 = z \qquad (3\text{-}20)$$

and denoted partial derivatives by

$$u_x = \frac{\partial u}{\partial x} , \qquad u_y = \frac{\partial u}{\partial y} , \qquad \text{etc.}$$

The matrix $\underline{U}$ can be split up into a symmetric part $\underline{E}$ and an antisymmetric part $\underline{R}$:

$$\underline{U} = \underline{E} + \underline{R} , \qquad (3\text{-}21)$$

where

$$\underline{E} = [e_{ij}] , \qquad e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad (3\text{-}22)$$

is the <u>strain tensor</u> (see also sec. 5) and

$$\underline{R} = [r_{ij}] , \qquad r_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \qquad (3\text{-}23)$$

is the (infinitesimal) <u>rotation tensor</u>.

For the solution (3-12) we have

$$\underline{U} = \begin{bmatrix} 0 & 0 & \beta\cos\sigma t \\ 0 & 0 & \beta\sin\sigma t \\ (\nu - \beta)\cos\sigma t & (\nu - \beta)\sin\sigma t & 0 \end{bmatrix} , \qquad (3\text{-}24)$$

$$\underline{E} = \begin{bmatrix} 0 & 0 & \frac{1}{2}\nu\cos\sigma t \\ 0 & 0 & \frac{1}{2}\nu\sin\sigma t \\ \frac{1}{2}\nu\cos\sigma t & \frac{1}{2}\nu\sin\sigma t & 0 \end{bmatrix} , \qquad (3\text{-}25)$$

$$\underline{R} = \begin{bmatrix} 0 & 0 & (\beta - \frac{1}{2}\nu)\cos\sigma t \\ 0 & 0 & (\beta - \frac{1}{2}\nu)\sin\sigma t \\ -(\beta - \frac{1}{2}\nu)\cos\sigma t & -(\beta - \frac{1}{2}\nu)\sin\sigma t & 0 \end{bmatrix} . \qquad (3\text{-}26)$$

The comparison with (3-14), (3-15), and (3-16) shows that

$$\underline{u}' = \underline{T}_1^1 = \underline{R}\,\underline{x} \;, \qquad \underline{u}'' = \underline{S}_2^1 = \underline{E}\,\underline{x} \;. \qquad (3\text{-}27)$$

The symmetric strain tensor $\underline{E}$, which characterizes an affine deformation of the volume element, is thus seen to be related to the

spheroidal mode (2,1), and the antisymmetric rotation tensor $\underline{R}$ is related to the toroidal mode (1,1), which thus represents an (infinitesimal) rotation.

The amplitude of this rotation, as a function of r, is represented by $T(r)$, whereas $H(r)$ characterizes radial deformation and $L(r)$, tangential deformation (displacement normal to the radius vector). This is immediately evident from (3-2).

4. Method of Poincaré - Jeffreys

Poincaré (1910, sec. II) expressed the coordinates x, y, z of a fluid particle as a linear function of its coordinates x_0, y_0, z_0 in a standard position:

$$x = a_{11}(t)x_0 + a_{12}(t)y_0 + a_{13}(t)z_0 \;,$$
$$y = a_{21}(t)x_0 + a_{22}(t)y_0 + a_{23}(t)z_0 \;, \qquad (4\text{-}1)$$
$$z = a_{31}(t)x_0 + a_{32}(t)y_0 + a_{33}(t)z_0 \;,$$

or briefly

$$x_i = a_{ij}(t)x_{0i} \;, \qquad (4\text{-}2)$$

using the notation (3-20), the summation convention implying summation over a subscript occurring twice (in our case, j).

Jeffreys (1949) used this method, employing the "nutation frame" $x_1^o x_2^o x_3^o$ (sec. 1). Thus we should have to write x_i^o instead of x_i , and x_{0i}^o instead of x_{0i} , but we shall take the superscript 0 for granted. Thus, x_{0i} are the coordinates of a liquid particle in an "undisturbed reference state", in which the liquid is at rest in the nutation frame (rotating uniformly with respect to inertial space).

Then the displacement

$$\underline{u} = \underline{x} - \underline{x}_0 \qquad (4\text{-}3)$$

will be small, and (4-2) gives

$$u_i = x_i - x_{0i} = a_{ij}x_{0j} - x_{0i}$$

or

$$u_i = \alpha_{ij}x_{0j} \tag{4-4}$$

with

$$\alpha_{ij} = a_{ij} - \delta_{ij} , \tag{4-5}$$

the Kronecker delta δ_{ij} denoting the elements of the unit matrix as usual.

The solution (2-21) is clearly of this linear form. Generally, the linear representation (4-4) is seen to be related to a quadratic pertu: ng potential V_e , a quadratic pressure p_1 , and a quadratic function ψ as (2-12) shows. In terms of spherical harmonics, it corresponds to perturbations which are harmonics of the second degree.

The transition from the standard position $\underline{x}_0$ to the actual posit on $\underline{x}$ may be described in the following way (Fig. 4.1). First, there is a dilatation which transforms the elliptical core into a sphere it represented by the matrix

$$\underline{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{a}{c} \end{bmatrix} , \tag{4-6}$$

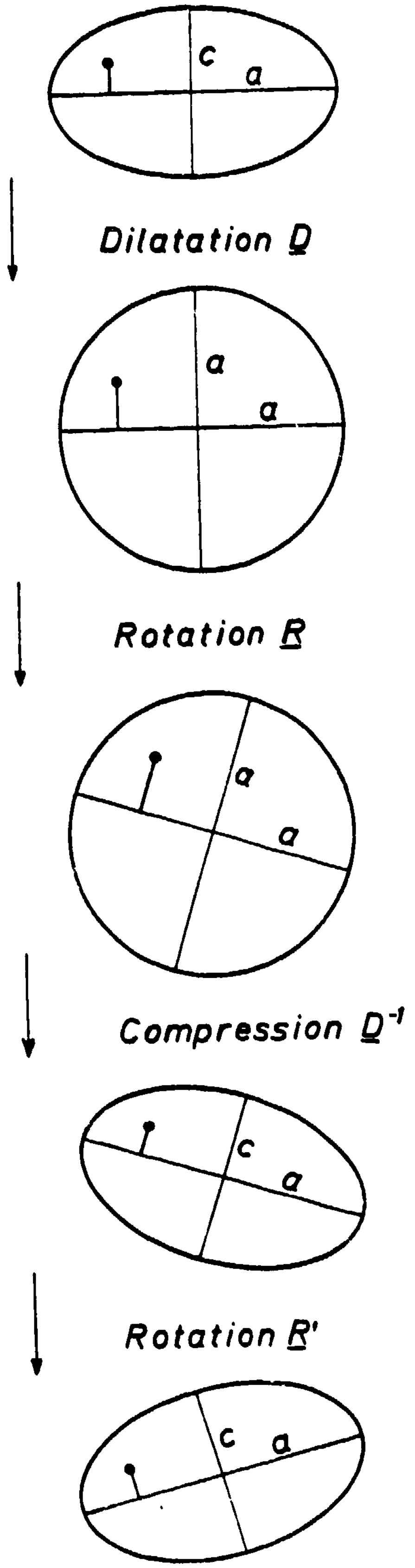

FIGURE 4.1. Method of Jeffreys

where a and c denote the semiaxes of the core, considered as an ellipsoid of revolution. The next step is a small rotation by the matrix

$$\underline{R} = \underline{I} + \begin{bmatrix} 0 & -n & l \\ n & 0 & m \\ -l & -m & 0 \end{bmatrix} = \underline{I} + \underline{F} \quad , \qquad (4\text{-}7)$$

which differs from the identity $\underline{I}$ by the small antisymmetric matrix $\underline{F}$. The compression by the matrix $\underline{D}^{-1}$ transforms the sphere back into our ellipsoid, and a final rotation

$$\underline{R}' = \underline{I} + \begin{bmatrix} 0 & -n' & l' \\ n' & 0 & m' \\ -l' & -m' & 0 \end{bmatrix} = \underline{I} + \underline{F}' \qquad (4\text{-}8)$$

gives the actual position (m' has no relation to m , etc.).

The composition of these linear transformations gives

$$\underline{x} = \underline{R}'\underline{D}^{-1}\underline{R}\,\underline{D}\,\underline{x}_0 = (\underline{I} + \underline{\Delta})\underline{x}_0 = \underline{x}_0 + \underline{\Delta}\,\underline{x}_0 \quad , \qquad (4\text{-}9)$$

where the small matrix $\underline{\Delta}$ is expressed by

$$\underline{\Delta} = (\underline{I} + \underline{F}')\underline{D}^{-1}(\underline{I} + \underline{F})\underline{D} - \underline{I} \; . \tag{4-10}$$

Retaining only terms linear in $\underline{F}$ and $\underline{F}'$ we get

$$\underline{\Delta} = [\alpha_{ij}] = \underline{F}' + \underline{D}^{-1}\underline{F}\,\underline{D} \; ; \tag{4-11}$$

in fact, (4-3) and (4-9) give

$$\underline{u} = \underline{\Delta}\,\underline{x}_0 \; , \tag{4-12}$$

so that the comparison with (4-4) shows that the matrix $\underline{\Delta}$ has the elements α_{ij} .

Working out the matrix multiplications in (4-11) we get, excluding quadratic and higher terms,

$$\begin{aligned}
&\alpha_{11} = \alpha_{22} = \alpha_{33} = 0 \; , \\
&\alpha_{12} = -(n + n') \; , \qquad && \alpha_{21} = n + n' \; , \\
&\alpha_{13} = \frac{a}{c}l + l' \; , && \alpha_{31} = -\left(\frac{c}{a}l + l'\right) \; , \\
&\alpha_{23} = \frac{a}{c}m + m' \; , && \alpha_{32} = -\left(\frac{c}{a}m + m'\right) \; .
\end{aligned} \tag{4-13}$$

Note that the vectors $\underline{x}, \underline{x}$, $\underline{u}$ refer to the nutation frame.

Equations of motion. Let us use the equation of motion (1-13) for the nutation frame:

$$\ddot{u} - 2\Omega\dot{v} = -\frac{\partial\psi}{\partial x} ,$$
$$\ddot{v} + 2\Omega\dot{u} = -\frac{\partial\psi}{\partial y} , \qquad (4\text{-}14)$$
$$\ddot{w} = -\frac{\partial\psi}{\partial z} ,$$

and substitute Jeffreys' representation

$$u_i = \alpha_{ij}x_{oj} , \qquad (4\text{-}15)$$

in which, of course, $u_1 = u$, $u_2 = v$, $u_3 = w$. We have, for instance,

$$u = \alpha_{1j}x_{oj} = \alpha_{11}x_o + \alpha_{12}y_o + \alpha_{13}z_o , \qquad (4\text{-}16)$$

and the differentiation gives

$$\ddot{u} = \ddot{\alpha}_{1j}x_{oj} \qquad (4\text{-}17)$$

since the reference position x_{oj} does not change with time.

Thus the system (4-14) becomes:

$$\ddot{\alpha}_{1j}x_{oj} - 2\Omega\dot{\alpha}_{2j}x_{oj} = -\frac{\partial\psi}{\partial x} ,$$
$$\ddot{\alpha}_{2j}x_{oj} + 2\Omega\dot{\alpha}_{1j}x_{oj} = -\frac{\partial\psi}{\partial y} , \qquad (4\text{-}18)$$
$$\ddot{\alpha}_{3j}x_{oj} = -\frac{\partial\psi}{\partial z} .$$

Since α_{ik} and its derivatives are small, we may put

$$x_{01} \doteq x_1 = x \; ,$$
$$x_{02} \doteq x_2 = y \; , \qquad (4\text{-}19)$$
$$x_{03} \doteq x_3 = z$$

in these equations. Let us now differentiate the first equation of (4-18) with respect to z and the third, with respect to x. In view of (4-19) we get

$$\ddot{\alpha}_{13} - 2\Omega\dot{\alpha}_{23} = - \frac{\partial^2\psi}{\partial x \partial z} \; ,$$
$$\ddot{\alpha}_{31} \qquad = - \frac{\partial^2\psi}{\partial x \partial z} \; . \qquad (4\text{-}20)$$

We subtract both equations, obtaining

$$\ddot{\alpha}_{13} - \ddot{\alpha}_{31} - 2\Omega\dot{\alpha}_{23} = 0 \; . \qquad (4\text{-}21)$$

From the second and third equation of (4-18) we get in the same way

$$\ddot{\alpha}_{23} - \ddot{\alpha}_{32} + 2\Omega\dot{\alpha}_{13} = 0 \; . \qquad (4\text{-}22)$$

(4-21) and (4-22) represent equations of motion in terms of matrix elements α_{ij} .

In the remaining part of this section we shall compare results of the methods of Poincaré, Jeffreys, and Molodensky. To make essential relations transparent, we shall limit ourselves to Poincaré's model: the mantle is rigid, the earth's surface and the core-mantle

boundary are concentric and coaxial ellipsoids of revolution, and the core consists of a homogeneous and incompressible liquid. This is the model considered in (Moritz, 1980b, secs. 12 and 13).

Relations between body frame and nutation frame. The coordinates of any point in the body frame, forming the vector $\underline{x}$, and of the same point in the nutation frame, forming the vector $\underline{x}^o$, are related by (1-17) and (1-18) :

$$\underline{x} = (\underline{I} + \underline{\Theta})\underline{x}^o \ , \tag{4-23}$$

$$\underline{\Theta} = \begin{bmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{bmatrix} . \tag{4-24}$$

Thus $\underline{I} + \underline{\Theta}$ rotates from the nutation frame to the body frame, whereas the matrix (4-8) rotates the body frame represented by the axes of the ellipsoid) to the inertial frame. Thus

$$\underline{F}' = -\underline{\Theta} \ , \tag{4-25}$$

and the comparison between (4-8) and (4-24) gives

$$\theta_1 = -m' \ , \qquad \theta_2 = l' \ . \tag{4-26}$$

We shall assume

$$\theta_3 = n' = 0 \tag{4-27}$$

for polar motion (Moritz, 1980b, p. 85).

We introduce the complex quantity

$$w = \theta_1 + i\theta_2 \tag{4-28}$$

(<u>ibid.</u>, p. 80), and define similarly

$$\lambda' = l' + im' \ . \tag{4-29}$$

Then (4-26) is equivalent to the simple complex equation

$$w = i\lambda' \ . \tag{4-30}$$

A geometrical interpretation of the complex number λ' is obtained by considering the nutation n_F of the figure axis (our x_3 axis). We have (<u>ibid.</u>, p. 128) :

$$n_F = -iw \ ,$$

and by (4-30),

$$n_F = \lambda' \ . \tag{4-31}$$

Thus λ' represents simply the nutation of the figure axis.

Let us now consider the direction of the rotation axis in

the body frame, which represents nothing else than polar motion:

$$p_R = m_1 + im_2 \; ; \tag{4-32}$$

cf. (ibid., pp. 93 and 129). By (1-19a,b) we have

$$\underline{\omega} = \underline{\omega}_0 + \delta\underline{\omega} = \begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix} + \begin{bmatrix} \omega_1 \\ \omega_2 \\ 0 \end{bmatrix} \tag{4-33}$$

in view of (1-19c), where

$$\omega_1 = \Omega m_1 \; , \qquad \omega_2 = \Omega m_2 \; . \tag{4-34}$$

In agreement with (ibid., p. 107) we put

$$u = \omega_1 + i\omega_2 \; , \tag{4-35}$$

so that

$$u = \Omega(m_1 + im_2) = \Omega p_R \quad . \tag{4-36}$$

The relation between u and w is

$$u = \dot{w} + i\Omega w \tag{4-37}$$

(*ibid.*, p. 115), and for w proportional to $e^{i\sigma t}$ as usual,

$$w = w_0 e^{i\sigma t} \qquad (w_0 = \text{const.}) \tag{4-38}$$

this becomes

$$u = i(\sigma + \Omega) w \ . \tag{4-39}$$

Together with (4-36) and (4-30) this gives

$$p_R = i \frac{\sigma + \Omega}{\Omega} w = - \frac{\sigma + \Omega}{\Omega} \lambda' \ , \tag{4-40}$$

in agreement with *ibid.*, eq. (13-38).

By (1-28) we have

$$p_R = \epsilon (\cos\sigma t + i \sin\sigma t) = \epsilon e^{i\sigma t} \ , \tag{4-41}$$

so that

$$\epsilon e^{i\sigma t} = - \frac{\sigma + \Omega}{\Omega} \lambda' = - \frac{\sigma + \Omega}{\Omega} (l' + i m') \tag{4-42}$$

represents a relation between Molodensky's parameter ϵ and Jeffreys' parameters l' and m' .

Relation between Jeffreys' and Poincaré's methods. Let us substitute (4-13) into (4-21) and (4-22), taking account of

the fact that a and c are constant for a rigid mantle. The result is

$$\left(\frac{a}{c} + \frac{c}{a}\right) \ddot{l} + 2\ddot{l}' - 2\Omega \frac{a}{c} \dot{m} - 2\Omega\dot{m}' = 0$$
$$\left(\frac{a}{c} + \frac{c}{a}\right) \ddot{m} + 2\ddot{m}' + 2\Omega \frac{a}{c} \dot{l} + 2\Omega\dot{l}' = 0 \ . \tag{4-43}$$

These are the equations of motion for Jeffreys' parameters.

It is easy to bring them to a form familiar from (Moritz, 1980b, sec. 12). By (*ibid.*, p. 103) we have for rotational symmetry ($a = b$)

$$A_c = \frac{1}{5} M_c(a^2 + c^2) \ ,$$
$$C_c = \frac{2}{5} M_c a^2 \ , \tag{4-44}$$
$$F = \frac{2}{5} M_c ac \ ,$$

M_c denoting the mass of the core and A_c and C_c its principal moments of inertia; F is an auxiliary quantity. Then, on multiplying them by $M_c ac/5$, eqs. (4-43) may be written:

$$A_c\ddot{l} + F\ddot{l}' - C_c\Omega\dot{m} - F\Omega\dot{m}' = 0 \ ,$$
$$A_c\ddot{m} + F\ddot{m}' + C_c\Omega\dot{l} + F\Omega\dot{l}' = 0 \ . \tag{4-45}$$

This system has been used by Jeffreys (1949, p. 677, eqs. (13) and (14)).

It is similar, though not identical, to Poincaré's equations (12-19) in (Moritz, 1980b, p.107). It is, however, identical to a related equation, namely to the second equation of the system (12-46), ibid., p. 115.

To see this, let us go to complex notation, using (4-29) and, similarly,

$$\lambda = l + im \quad . \tag{4-46}$$

Then the system (4-45) is equivalent to the single complex equation

$$A_c\ddot{\lambda} + F\ddot{\lambda}' + iC_c\Omega\dot{\lambda} + iF\Omega\dot{\lambda}' = 0 \quad . \tag{4-47}$$

Now we use (4-30) and put, similarly,

$$v = i\dot{\lambda} \quad . \tag{4-48}$$

Then (4-47) becomes

$$F(\ddot{w} + i\Omega\dot{w}) + A_c\dot{v} + iC_c\Omega v = 0 \quad , \tag{4-49}$$

identical to the second equation of (12-46), ibid.

The fact that we have arrived at this equation in a completely different way, serves as a check and shows a relation between Jeffreys' (1949) method and "Poincaré's first method" (Poincaré, 1910, sec. II), of which Jeffreys' method is a modification and which thus is quite similar, except that Poincaré uses $\alpha_{13}, \alpha_{31}, \alpha_{23}, \alpha_{32}$ instead of l, l', m, m' ; these parameters are related by (4-13). The equations of motion for Poincaré's parameters are (4-21) and (4-22).

<u>Relation between Molodensky's and Poincaré's methods</u>.

Using the complex numbers u and v, given by (4-35) and (4-48), we may write Poincaré's basic equations in the form (Moritz, 1980b, eq. (12-22); we take $L = 0$) :

$$A\dot{u} + F\dot{v} - i(C - A)\Omega u + iF\Omega v = 0 , \tag{4-50}$$

$$F\dot{u} + A_c\dot{v} + iC_c\Omega v = 0 , \tag{4-51}$$

A and C denoting the principal moments of inertia of the whole earth.

With

$$u = u_0 e^{i\sigma t} , \qquad v = v_0 e^{i\sigma t} \tag{4-52}$$

and

$$F = A_c$$

(<u>ibid.</u>, p. 109) this becomes

$$[A\sigma - (C - A)\Omega]u_0 + A_c(\sigma + \Omega)v_0 = 0 , \tag{4-53}$$

$$A_c\sigma u_0 + (A_c\sigma + C_c\Omega)v_0 = 0 . \tag{4-54}$$

We shall now show that Poincaré's equation (4-54) is a simple consequence of Molodensky's theory applied to the Poincaré model.

We use Molodensky's solution in the form (3-12) :

$$u_1 = \beta x_3 \cos\sigma t \quad ,$$
$$u_2 = \beta x_3 \sin\sigma t \quad ; \qquad (4\text{-}55)$$

The third equation we shall not need. These equations refer to the body frame.

These equations may be compared to (4-4). There, however, the nutation frame is used. It differs from the body frame by the final rotation (4-8). If this rotation were not performed, we should end up in the body frame. Since this is desired, we just do not perform the final rotation or simply put $l' = m' = n' = 0$. Then (4-4) with (4-13) gives

$$u_1 = \frac{a}{c}\, l x_3 \quad ,$$
$$u_2 = \frac{a}{c}\, m x_3 \quad ; \qquad (4\text{-}56)$$

we have identified x_{03} on the right-hand side with x_3 , which is permissible because of the smallness of l and n .

The comparison between (4-55) and (4-56) gives immediately

$$l = \frac{c}{a}\,\beta\cos\sigma t \; , \qquad m = \frac{c}{a}\,\beta\sin\sigma t \; , \qquad (4\text{-}57)$$

or, with (4-46),

$$\lambda = \frac{c}{a}\,\beta e^{i\sigma t}. \qquad (4\text{-}58)$$

By (4-48) we get

$$v = -\sigma \frac{c}{a} \beta \, e^{i\sigma t} \, , \tag{4-59}$$

and by (4-52)

$$v_0 = -\sigma \frac{c}{a} \beta \, , \tag{4-60}$$

which is a relation between Poincaré's variable v and Molodensky's parameter β .

From (4-36) and (4-41) we get an analogous relation

$$u_0 = \Omega \epsilon \; . \tag{4-61}$$

Thus Molodensky's parameters ϵ and β are seen to be basically equivalent to u and v .

Let us now consider eq. (2-59) for Molodensky's theory. Since the core-mantle boundary is assumed to be rigid, there is no normal displacement:

$$q = 0 \; . \tag{4-62}$$

Thus (2-59) gives simply

$$\nu - e^2 \beta = 0 \; ; \tag{4-63}$$

we could have obtained this condition also from (2-65). It will now be shown that (4-63) is equivalent to (4-54).

First, (2-58) and (4-63) give:

$$\frac{\sigma + \Omega}{\sigma} \beta - \frac{\Omega}{\sigma} \epsilon = \frac{1}{2} e^2 \beta$$

or

$$\beta + \frac{\Omega}{\sigma}(\beta - \varepsilon) = \frac{1}{2} e^2 \beta \quad . \tag{4-64}$$

By definition,

$$e^2 = \frac{a^2 - c^2}{a^2} , \tag{4-65}$$

and the comparison of eqs. (12-29) and (12-30) of (Moritz, 1980b, p. 109) shows that

$$\frac{1}{2} e^2 = \frac{C_c - A_c}{C_c} \quad . \tag{4-66}$$

This is substituted into (4-64). By elementary algebraic manipulations this is brought into the form

$$A_c \sigma \varepsilon + (A_c \sigma + C_c \Omega)(\beta - \varepsilon) = 0 \ . \tag{4-67}$$

Let us finally bring (4-60) into a different form. Using

$$\frac{c}{a} = \sqrt{1 - e^2} \doteq 1 - \frac{1}{2} e^2 \tag{4-68}$$

we get:

$$-v_0 = \sigma\beta - \frac{1}{2}\sigma e^2\beta$$

$$= \sigma\beta - \frac{1}{2}\sigma\nu \qquad \text{(by (4-63))}$$

$$= \sigma\beta - [(\sigma + \Omega)\beta - \Omega\varepsilon] \qquad \text{(by (2-58))}$$

which gives simply

$$v_0 = \Omega(\beta - \varepsilon) \ .$$

Now (4-61) and (4-69) are substituted in (4-67) and immediately give (4-54), which was to be shown.

In sec. 10 we shall in a similar way derive Poincaré's equation (4-53) from Molodensky's theory, completing the proof of equivalence of both theories for the Poincaré model.

PART B

THEORIES OF MOLODENSKY TYPE

5. Equations of Elasticity

The strain tensor. We briefly summarize basic formulas from elasticity theory, for details referring the reader to any textbook on theoretical physics or to special treatises such as (Love, 1927) or (Green and Zerna, 1968).

Let $\underline{x}_0$ and $\underline{x}$ denote the position vectors of a point of an elastic body before and after deformation (Fig. 5.1). Then the difference

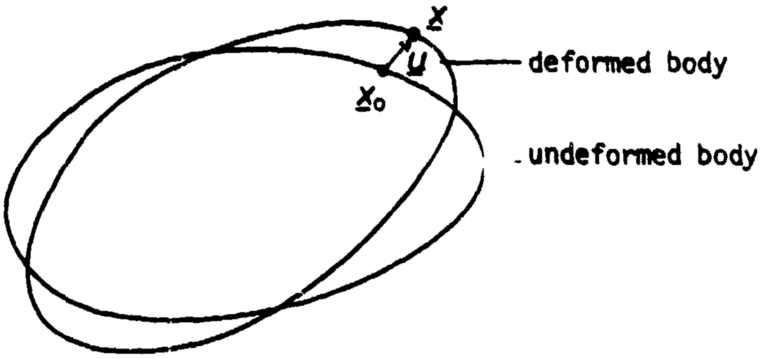

FIGURE 5.1. Elastic deformation.

$$\underline{u} = \underline{x} - \underline{x}_0 \tag{5-1}$$

is the displacement vector, or deformation vector. Denoting the components of $\underline{u}$ by u_i as usual, we get the matrix of deformation gradients

$$\underline{U} = \left[\frac{\partial u_i}{\partial x_j}\right] \quad ; \tag{5-2}$$

in view of the smallness of u_i it is irrelevant, to the linear approximation, whether we differentiate with respect to x_{0i} or to x_j .

The symmetric part of $\underline{U}$,

$$\underline{E} = \frac{1}{2}\left(\underline{U} + \underline{U}^T\right) = [e_{ij}] \quad , \tag{5-3}$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad , \tag{5-4}$$

is the strain tensor, and the antisymmetric part of $\underline{U}$,

$$\underline{R} = \frac{1}{2}\left(\underline{U} - \underline{U}^T\right) = [r_{ij}] \quad , \tag{5-5}$$

$$r_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \quad , \tag{5-6}$$

is the (infinitesimal) rotation tensor. All this is fully ana-

logous to sec. 3.

The rotation tensor $\underline{R}$ plays rather a small role in elasticity. Of basic importance is the strain tensor $\underline{E}$. Denote a line element before and after deformation by ds_0 and ds , and a volume element, by dv_0 and dv . Then the change of ds is given by

$$ds^2 - ds_0^2 = 2d\underline{x}^T\underline{E}d\underline{x} \quad , \tag{5-7}$$

and the volume dilatation Θ by

$$\Theta = \frac{dv - dv_0}{dv_0} = \text{trace } \underline{E} = e_{11} + e_{22} + e_{33} =$$

$$= \text{div } \underline{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \quad . \tag{5-8}$$

Hence the change of both length and volume can be expressed in terms of the tensor $\underline{E}$.

We shall now again use the Einstein summation convention: if an index occurs twice in a product or a similar expression, summation with respect to that index is automatically implied. For instance,

$$a_j b_j = \sum_{j=1}^{3} a_j b_j = a_1 b_1 + a_2 b_2 + a_3 b_3 \ . \tag{5-9}$$

Thus,(5-7) may be written

$$ds^2 - ds_0^2 = 2e_{ij}dx_i dx_j \ , \tag{5-10}$$

and (5-8) reduces to

$$\Theta = e_{kk} = \frac{\partial u_k}{\partial x_k} \; . \tag{5-11}$$

<u>The stress tensor.</u> There are two basic tensors[1] in elasticity theory. The first is the strain tensor $\underline{E}$, the second the <u>stress tensor</u>, denoted by

$$\underline{P} = [p_{ij}] \; . \tag{5-12}$$

This tensor is characterized by the fact that the stress vector $\underline{p}$ on any surface element normal to a unit vector $\underline{n}$ is expressed by

$$\underline{p} = \underline{P}\,\underline{n} \tag{5-13}$$

or

$$p_i = p_{ij} n_j \; . \tag{5-14}$$

The tensor $\underline{P}$ is symmetric:

$$\underline{P}^T = \underline{P} \; , \qquad p_{ji} = p_{ij} \; . \tag{5-15}$$

---

1) A tensor may be viewed simply as a matrix having physical meaning, such as inertia tensor, strain tensor, or stress tensor. Mathematicians, of course, characterize a tensor in terms of its behavior with respect to coordinate transformations, but this will not be needed here.

An important special case is that of $\underline{P}$ being diagonal and proportional to the unit matrix $\underline{I}$:

$$\underline{P} = -p\underline{I} , \qquad p_{ij} = -p\delta_{ij} , \qquad (5\text{-}16)$$

δ_{ij} being the Kronecker delta already used in sec. 4 : $\delta_{ij} = 1$ or 0 according to whether $j = i$ or $j \neq i$; it denotes the elements of the unit matrix. The proportionality factor p is called hydrostatic pressure since the form (5-16) plays a special role in hydrodynamics.

Strain tensor $\underline{E}$ and stress tensor $\underline{P}$ are linearly related by Hooke's law:

$$p_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} \qquad (5\text{-}17)$$

or, by (5-11)

$$\underline{P} = \lambda\Theta\underline{I} + 2\mu\underline{E} , \qquad (5\text{-}18)$$

where λ and μ are Lamé's parameters. In terms of components we have, for instance,

$$\begin{aligned} p_{11} &= \lambda\Theta + 2\mu \frac{\partial u_1}{\partial x_1} , \\ p_{12} &= \mu \left(\frac{\partial u}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) . \end{aligned} \qquad (5\text{-}19)$$

Equations of motion. They have the form

$$\rho \ddot{u}_i = \rho f_i + \frac{\partial p_{ij}}{\partial x_j} , \tag{5-20}$$

where ρ is the density, f_i is the body force per unit mass, and $\partial p_{ij}/\partial x_j$ represents the resultant force due to the stresses p_{ij} acting on a volume element. In vector notation we may write (5-20) as

$$\rho \ddot{\underline{u}} = \rho \underline{f} + \operatorname{div} \underline{P} . \tag{5-21}$$

The substitution of Hooke's law (5-17) gives

$$\rho \ddot{u}_i = (\lambda + \mu) \frac{\partial \Theta}{\partial x_i} + \mu \Delta u_i + \rho f_i , \tag{5-22}$$

or

$$\rho \ddot{\underline{u}} = (\lambda + \mu) \operatorname{grad} \Theta + \mu \Delta \underline{u} + \rho \underline{f} , \tag{5-23}$$

Θ being the volume dilatation (5-11) and

$$\Delta u_i = \frac{\partial^2 u_i}{\partial x_j \partial x_j} = \frac{\partial^2 u_i}{\partial x_1^2} + \frac{\partial^2 u_i}{\partial x_2^2} + \frac{\partial^2 u_i}{\partial x_3^2} . \tag{5-24}$$

denoting Laplace's operator applied to the displacement component u_i .

Relation to hydrodynamics. Hydrostatic fluid pressure has the form (5-16). The comparison of (5-16) and (5-18) shows that,

for this case,

$$\mu = 0 \;, \qquad \lambda\Theta = -p \;. \tag{5-25}$$

Then (5-23) reduces to

$$\rho\ddot{\underline{u}} = -\mathrm{grad}\, p + \rho\underline{f}$$

or

$$\ddot{\underline{u}} = \underline{f} - \rho^{-1}\mathrm{grad}\, p \;, \tag{5-26}$$

which is (1-1) with $\ddot{\underline{u}} = \ddot{\underline{X}}$ and $\underline{f} = \mathrm{grad}\, V$.

Thus we have shown the basic fact that the equations of hydrodynamics may formally be considered a special case of the equations of elasticity if the Lamé constants are replaced by (5-25). For incompressible fluids, there is $\Theta = 0$ but p finite, so that for them we must take

$$\mu = 0 \qquad \text{and} \qquad \lambda \to \infty \tag{5-27}$$

in such a way that

$$\lim(\lambda\Theta) = -p \tag{5-28}$$

remains finite.

6. Application to the Earth

In order to apply the elastic equations of motion (5-20) or (5-21) to the rotating earth, we must transform them to a system xyz rotating with the constant angular velocity vector $\underline{\omega}$. In this system we have as in sec. 1 :

$$\begin{aligned} \underline{x} &= \underline{x}_0 + \underline{u} , \\ \dot{\underline{x}} &= \dot{\underline{u}} , \\ \ddot{\underline{x}} &= \ddot{\underline{u}} , \end{aligned} \tag{6-1}$$

$\underline{x}_0$ denoting an undisturbed equilibrium state corresponding to the uniform rotation. Then, in complete analogy to (1-4), eq. (5-21) is replaced by

$$\rho[\ddot{\underline{u}} + 2\underline{\omega} \times \dot{\underline{u}} + \underline{\omega} \times (\underline{\omega} \times \underline{x})] = \rho \,\mathrm{grad}\, V + \mathrm{div}\, \underline{P} . \tag{6-2}$$

The comparison of the left-hand sides of (1-4) and (6-2) shows that we have used (6-1) and

$$\dot{\underline{\omega}} = 0 \tag{6-3}$$

for uniform rotation. Note that this means that the <u>coordinate system</u> xyz , not the earth, rotates uniformely: small non-uniformities of the rotation of an elastic earth are included in the vector $\underline{u}$.

Furthermore we have put

$$\underline{f} = \operatorname{grad} V \quad , \tag{6-4}$$

expressing the gravitational force as the gradient of the gravitational potential V. Since the angular velocity vector $\underline{\omega}$ is assumed constant, the rotation axis has an invariable direction in space, which we may take as the z-axis, so that (1-6) holds. Then the centrifugal force $\underline{\omega} \times (\underline{\omega} \times \underline{x})$ may be expressed as the gradient of the centrifugal potential (1-8) :

$$\Phi = \frac{1}{2} \Omega^2 (x^2 + y^2) \quad , \tag{6-5}$$

and (6-2) becomes

$$\rho(\ddot{\underline{u}} + 2\underline{\omega} \times \dot{\underline{u}}) = \rho \operatorname{grad} W + \operatorname{div} \underline{P} \, , \tag{6-6}$$

the gravity potential W being defined by

$$W = V + \Phi \tag{6-7}$$

as usual.

<u>The equilibrium state.</u> For equilibrium we have $\underline{u} \equiv 0$ since there is no relative movement. Denote the (in general variable) equilibrium density by ρ_0, and the corresponding potential by W_0. The stress tensor $\underline{P}_0$ is assimed to correspond to hydrostatic equilibrium, so that

$$\underline{P}_0 = -p_0 I \tag{6-8}$$

by (5-16); p_0 is also variable in general.

Then (6-6) reduces to

$$0 = \rho_0 \operatorname{grad} W_0 + \operatorname{div} \underline{P}_0 \quad , \tag{6-9}$$

or by (6-8)

$$0 = \rho_0 \operatorname{grad} W_0 - \operatorname{grad} p_0 \quad , \tag{6-10}$$

which is again the well-known condition for hydrostatic equilibrium (2-27).

Deviations from equilibrium. Let a material particle in

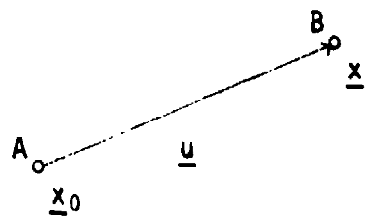

FIGURE 6.1. Displacement from equilibrium

the equilibrium state be at point A , having the position vector $\underline{x}$, and let the same material particle after deformation be at point B , having the position vector $\underline{x}$; cf. Fig. 6.1. All this is quite analogous to sec. 2; cf. Fig. 2.1.

Now the left-hand side of (6-6) is small together with $\underline{u}$, and so is the right-hand side

$$\underline{R} \equiv \rho \operatorname{grad} W + \operatorname{div} \underline{P} , \qquad (6\text{-}11)$$

so that it is irrelevant, to the first order, whether we take $\underline{R}$ at A or B . On the other hand, the single terms $\rho \operatorname{grad} W$ and $\operatorname{div} \underline{P}$ are not small, so that both must be taken at A or both at B , but consistently. We take them at point B . Thus

$$\underline{R} = \underline{R}_B = \rho_B \operatorname{grad} W_B + \operatorname{div} \underline{P}_B \quad . \qquad (6\text{-}12)$$

Now

$$\Phi_0 = \frac{1}{2}\,\Omega^2 (x_0^2 + y_0^2) ,$$

$$\Phi = \frac{1}{2}\,\Omega^2 (x^2 + y^2)$$

differ only by negligibly small quantities since the centrifugal potential itself is already small. Thus we may put $\Phi_0 = \Phi$, so that the potential disturbance is purely gravitational:

$$W - W_0 = (V + \Phi) - (V_0 + \Phi) = V - V_0$$

$$= V_e + V_1 , \qquad (6\text{-}13)$$

consisting of the tidal potential V_e and the change V_1 of the body's attractional potential caused by its change of shape due to the deformation; cf. (2-22).

We again denote the difference $W_B - W_A$ by η :

$$\eta = W_B - W_A = \underline{u} \cdot \operatorname{grad} W \doteq \underline{u} \cdot \operatorname{grad} W_0 =$$

$$= W_{0B} - W_{0A} \quad , \tag{6-14}$$

by (2-33). Thus (6-13) gives

$$W_B = W_{0B} + (V_e + V_1)_B \ ,$$

or

$$W_B = W_{0A} + \eta + V_e + V_1 \ , \tag{6-15}$$

using (6-14) and the fact that V_e and V_1 are so small that it is irrelevant whether we take them at A or B .

Let us now find a similar expression for ρ_B . During deformation, a mass element dM may change its shape but not its mass. Thus,

$$dM_B = dM_{0A} \ . \tag{6-16}$$

On the other hand, the corresponding volume element dv changes by

$$dv_B = (1 + \Theta) dv_{0A} \tag{6-17}$$

by (5-8), where

$$\Theta = \operatorname{div} \underline{u} \ . \tag{6-18}$$

Since $dM = \rho dv$, we have from (6-16) :

$$\rho_B dv_B = \rho_{0A} dv_{0A} \quad ,$$

and by (6-17)

$$\rho_B(1+\Theta) = \rho_{0A} \qquad (6\text{-}19)$$

or

$$\rho_B = \rho_{0A} - \rho_0\Theta \quad . \qquad (6\text{-}20)$$

Since Θ is very small, ρ_0 in the last term may refer to A or to B .

Let us now put

$$\rho_1 = \rho_B - \rho_{0B} = \rho_{1B} \doteq \rho_{1A} = \rho_A - \rho_{0A} \qquad (6\text{-}21)$$

for similar reasons as for (6-15). Then (6-20) gives

$$\rho_1 = \rho_B - \rho_{0B} = \rho_{0A} - \rho_0\Theta - \rho_{0B} \quad . \qquad (6\text{-}22)$$

Now,

$$\rho_{0B} = \rho_{0A} + \underline{u} \cdot \mathrm{grad}\rho_0 \qquad (6\text{-}23)$$

by Taylor's theorem. The combination of (6-22) and (6-23) finally gives

$$\rho_1 = -\rho_0\Theta - \underline{u} \cdot \mathrm{grad}\rho_0 \quad . \qquad (6\text{-}24)$$

This may be written in the form

$$\rho_1 = -\mathrm{div}(\rho_0 u) \ . \tag{6-25}$$

This formula is best derived using the index notation introduced in the preceding section. In fact, by (5-11),

$$\Theta = \frac{\partial u_k}{\partial x_k} \ , \tag{6-26}$$

$$\underline{u} \cdot \mathrm{grad}\rho_0 = u_k \frac{\partial \rho_0}{\partial x_k} \ , \tag{6-27}$$

so that (6-24) becomes

$$\rho_1 = -\rho_0 \frac{\partial u_k}{\partial x_k} - u_k \frac{\partial \rho_0}{\partial x_k} = - \frac{\partial(\rho_0 u_k)}{\partial x_k} \ , \tag{6-28}$$

which is nothing else than (6-25).

Note that

$$\begin{aligned} V_e + V_1 &= W_B - W_{0B} \ , \\ \rho_1 &= \rho_B - \rho_{0B} \end{aligned} \tag{6-29}$$

refer to the same point and are therefore <u>Eulerian increments</u> (sec. 2).

We also note that the change V_1 of the gravitational potential is produced by the density anomaly ρ_1 and is, therefore, related to it by Poisson's equation (Heiskanen and Moritz, 1967, p.5)

$$\Delta V_1 = -4\pi G\rho_1 = 4\pi G \mathrm{div}(\rho_0 \underline{u}) \qquad (6\text{-}30)$$

by (6-25). Here Δ denotes the Laplacian operator and G the gravitational constant.

Let us finally turn to the stress tensor $\underline{P}$. We define the incremental stress tensor $\underline{P}_1$ by

$$\underline{P}_1 = \underline{P}_B - \underline{P}_{0A} \; . \qquad (6\text{-}31)$$

It thus represents a Lagrangian increment since $\underline{P}$ and $\underline{P}_0$ refer to the same material particle but to different spatial points. This is quite analogous to (2-32).

Now we can substitute (6-15), (6-20), and (6-31) into (6-12):

$$\underline{R} = (\rho_{0A} - \rho_0\Theta)\mathrm{grad}(W_{0A} + n + V_e + V_1) + \mathrm{div}(\underline{P}_{0A} + \underline{P}_1)$$

$$= \rho_{0A}\mathrm{grad}W_{0A} + \mathrm{div}\underline{P}_{0A} -$$

$$-\rho_0\Theta\mathrm{grad}W_0 + \rho_0\mathrm{grad}(n + V_e + V_1) + \mathrm{div}\underline{P}_1 \quad .$$

We now take (6-9) at the point A and subtract it from the last equation. There remains

$$\underline{R} = -\rho_0\Theta\mathrm{grad}W_0 + \rho_0\mathrm{grad}(n + V_e + V_1) + \mathrm{div}\underline{P}_1 \; . \qquad (6\text{-}32)$$

By the definition (6-11), this is the right-hand side of the equation of motion (6-6), so that this equation becomes

$$\rho(\ddot{\underline{u}} + 2\underline{\omega} \times \dot{\underline{u}}) = -\rho\Theta \operatorname{grad} W + \rho \operatorname{grad}(\eta + V_e + V_1) + \operatorname{div}\underline{P}_1 . \tag{6-33}$$

Since now all summands are small of first order, we have been able to replace ρ_0 and W_0 by ρ and W, apart from second-order quantities. This simplifies the notation; still it is usually convenient to refer ρ and W in (6-33) to the equilibrium state.

Eq. (6-33) represents the basic equation of motion for our present purposes.

The elastic stress tensor $\underline{P}_1$. We note that $\underline{P}_1$ represents the elastic part of the stress tensor $\underline{P}$. In fact, $\underline{P}$ consists of a large hydrostatic part $\underline{P}_0$ which provides the huge pressure necessary to resist the gravitational compression, and a small residual part $\underline{P}_1$ which may be considered to be related to the small elastic deviations $\underline{u}$ by Hooke's law (5-17) or (5-18). Thus,

$$\underline{P}_1 = \lambda\Theta\underline{I} + 2\mu\underline{E} . \tag{6-34}$$

Putting

$$\underline{P}_1 = [p_{ij}] , \tag{6-35}$$

we may write

$$p_{ij} = \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) , \qquad (6\text{-}36)$$

in view of (5-4) and (5-17).

Various forms of the basic equation. The right-hand side R of the basic equation of motion (6-33) may be written in different forms. In this way it is seen that the motion equations used by different authors are equivalent in spite of their different forms.

It is convenient to use index notation. Then (6-32) becomes, on again omitting the subscript zero and using (6-14) and (6-26),

$$R_i = -\rho \frac{\partial u_k}{\partial x_k} \frac{\partial W}{\partial x_i} + \rho \frac{\partial u_k}{\partial x_i} \frac{\partial W}{\partial x_k} + \rho u_k \frac{\partial^2 W}{\partial x_i \partial x_k} +$$

$$+ \rho \frac{\partial}{\partial x_i} (V_e + V_1) + \frac{\partial p_{ij}}{\partial x_j} . \qquad (6\text{-}37)$$

This form has been given by Jeffreys and Vicente (1957a, p. 144) and used by Wahr (1981a). Smith (1974, p. 494) has

$$R_i = \frac{\partial p_{ij}}{\partial x_j} - \frac{\partial}{\partial x_i} \left(p_0 \frac{\partial u_k}{\partial x_k} \right) + \frac{\partial}{\partial x_k} \left(p_0 \frac{\partial u_k}{\partial x_i} \right) +$$

$$+ \rho \frac{\partial}{\partial x_i} (V_e + V_1) + \rho u_k \frac{\partial^2 W}{\partial x_i \partial x_k} , \qquad (6\text{-}38)$$

where p_0 is the hydrostatic pressure in (6-8). By differentiating the products, this is readily brought into the form

$$R_i = \frac{\partial p_{ij}}{\partial x_j} - \frac{\partial p_0}{\partial x_i}\frac{\partial u_k}{\partial x_k} + \frac{\partial p_0}{\partial x_k}\frac{\partial u_k}{\partial x_i} + \rho \frac{\partial}{\partial x_i}(V_e + V_1) +$$

$$+ \rho u_k \frac{\partial^2 W}{\partial x_i \partial x_k} \quad . \tag{6-39}$$

Since, by (6-10),

$$\frac{\partial p_0}{\partial x_i} = \rho_0 \frac{\partial W_0}{\partial x_i} \quad , \tag{6-40}$$

this is immediately seen to be equivalent to (6-37).

The concise form given by Dahlen (1972, p. 360):

$$R_i = \rho_1 \frac{\partial W}{\partial x_i} + \frac{\partial}{\partial x_i}(\rho n) + \rho \frac{\partial}{\partial x_i}(V_e + V_i) + \frac{\partial p_{ij}}{\partial x_j} \tag{6-41}$$

can be transformed to (6-37) in a similar way.

<u>Molodensky's equation.</u> Let us return to the form (6-33) :

$$\ddot{\underline{u}} + 2\underline{\omega} \times \underline{u} = \mathrm{grad}(V_e + V_1 + n) - \Theta \mathrm{grad} W + \rho^{-1} \mathrm{div} \underline{P}_1 \,. \tag{6-42}$$

It presupposes that the z-axis coincides with the rotation axis and that the rotation is uniform. In other terms, it holds for the <u>nutation frame</u> (sec. 1).

The corresponding equation of motion in the body frame is found as in sec. 1 : we must add (1-34) to the left-hand side, or subtract it from the right-hand side. The result is

$$\ddot{\underline{u}} + 2\underline{\omega} \times \underline{u} = \text{grad}\left(V_e + V_1 + \frac{\sigma + \Omega}{\Omega}\phi + \eta \right) -$$
$$- 2\,\frac{\sigma}{\Omega}\,\frac{\partial\phi}{\partial z}\,\underline{e}_3 - \Theta\,\text{grad}W + \frac{1}{\rho}\,\text{div}\underline{P}_1 \ , \qquad (6\text{-}43)$$

where $\underline{\omega} = (0,0,\Omega)$ as throughout the present section. This equation has been given by Molodensky (1961, eq.(11)); see also (Jobert, 1964, p. 66).

An incompressible fluid may formally be considered a special case of an elastic medium with $\Theta = 0$ and

$$\underline{P}_1 = -p_1 \underline{I} \ , \qquad (6\text{-}44)$$

p_1 being an incremental hydrostatic pressure, as we have seen at the end of the preceding section. For this particular case, (6-43) reduces to (2-1) with (2-38) and (2-39), as it should.

7. DIFFERENTIAL EQUATIONS FOR THE MANTLE

Molodensky's equation (6-43) holds for a rotating earth of arbitrary shape, in particular for an ellipsoidal earth. It furthermore holds for an elastic and for a liquid body, a liquid being formally considered a limiting case of an elastic body as we have seen at the end of sec. 5.

Eq. (6-43) also holds for a body that is partly elastic and partly liquid, such as for an earth consisting of an elastic mantle, a liquid outer core, and a solid elastic inner core, provided we add appropriate boundary conditions at the boundary surfaces between core and mantle and between outer and inner core.

Thus (6-43) may be considered a system of partial differential equations, to be solved for the displacement vector $\underline{u}$ and for other quantities. Such a uniform treatment of (6-43), together with the boundary conditions mentioned, for the whole earth is particularly suited for the use of a modern computer; it has been employed by Shen and Mansinha (1976) and, most recently and satisfactorily, by Wahr (1979); see sec. 11.

Molodensky (1961), however, persued a largely analytic approach, as did Jeffreys and Vicente (1957a,b) before him. Here it proves appropriate to treat core and mantle in a completely separate way. The core is treated by methods described in Part A of the present report, whereas for the mantle eqs. (6-43) are used, but in a greatly simplified form.

Spherical approximation. In fact, the consideration of ellipticity is essential, above all, for the core as we have seen in sec. 2. For the mantle, a spherical approximation is sufficient as

has been recognized already by Jeffreys (1949).

Molodensky neglects the flattening, the rotational effects, and even the free elastic vibrations of the mantle. He thus puts, formally,

$$\Omega = 0 , \qquad \Phi = 0 , \qquad \ddot{\underline{u}} = 0 \qquad (7\text{-}1)$$

(only for the mantle). These neglections, surprising as they are at first sight, can be justified by estimating orders of magnitude and, above all, by good agreement with the more precise theory. An essential simplification is achieved in this way.

Thus (6-43) reduces to

$$0 = \rho \operatorname{grad}(V_e + V_1 + \eta) - \rho\Theta \operatorname{grad} W + \operatorname{div}\underline{P}_1 \quad . \qquad (7\text{-}2)$$

In the sequel we shall more or less follow (Jobert, 1964).

Spherical coordinates. For a spherical earth it is appropriate to use spherical coordinates r (radius vector), θ (polar distance), and λ (longitude); cf. (2-41).

In order to express (7-2) in terms of r,θ,λ, we first note that the gradient of a function F has, in these coordinates, the components

$$\operatorname{grad} F = \begin{bmatrix} \dfrac{\partial F}{\partial r} \\[2ex] \dfrac{1}{r}\dfrac{\partial F}{\partial \theta} \\[2ex] \dfrac{1}{r\sin\theta}\dfrac{\partial F}{\partial \lambda} \end{bmatrix} , \qquad (7\text{-}3)$$

whereas in rectangular coordinates it has the components

$$\operatorname{grad} F = \begin{bmatrix} \dfrac{\partial F}{\partial x} \\ \dfrac{\partial F}{\partial y} \\ \dfrac{\partial F}{\partial z} \end{bmatrix} . \tag{7-4}$$

More difficult is the expression of the components of the stress tensor $\underline{P}_1$ in spherical coordinates. From standard texts (e.g., Love, 1927, pp. 56, 91; McConnell, 1931, p. 311) we get for the stress-strain equations (6-36) in coordinates r, θ, λ :

$$\begin{aligned}
p_{rr} &= \bar{\lambda}\Theta + 2\mu \frac{\partial u_r}{\partial r} , \\
p_{\theta\theta} &= \bar{\lambda}\Theta + 2\mu \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) , \\
p_{\lambda\lambda} &= \bar{\lambda}\Theta + 2\mu \left(\frac{1}{r\sin\theta} \frac{\partial u_\lambda}{\partial \lambda} + \frac{u_\theta}{r} \cot\theta + \frac{u_r}{r} \right) , \\
p_{r\theta} &= \frac{\mu}{r} \left(\frac{\partial u_r}{\partial \theta} + r \frac{\partial u_\theta}{\partial r} - u_\theta \right) , \\
p_{r\lambda} &= \frac{\mu}{r} \left(\frac{1}{\sin\theta} \frac{\partial u_r}{\partial \lambda} + r \frac{\partial u_\lambda}{\partial r} - u_\lambda \right) ,
\end{aligned} \tag{7-5}$$

$$p_{\theta\lambda} = \frac{\mu}{r}\left(\frac{\partial u_\lambda}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial u_\theta}{\partial\lambda} - u_\lambda\cot\theta\right) . \qquad (7\text{-}5)$$

Here u_r, u_θ, u_λ denote the spherical components of the displacement vector $\underline{u}$, and similarly p_{rr}, $p_{r\theta}$, etc. denote the components of the stress tensor $\underline{P}_1$. The Lamé parameters, formerly designated by λ, μ, are not denoted by $\bar{\lambda}$, μ, in order to avoid confusion with longitude λ .

These rather complicated-looking equations become immediately clear if we write (6-36) in rectangular coordinates xyz : putting $p_{xx} = p_{11}$, $p_{xy} = p_{12}$, etc., we have

$$p_{xx} = \bar{\lambda}\Theta + 2\mu\frac{\partial u_x}{\partial x} ,$$

$$p_{xy} = \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) , \quad \text{etc.} \qquad (7\text{-}6)$$

Thus there correspond

$$\partial x,\ \partial y,\ \partial z \quad \text{and} \quad \partial r,\ r\partial\theta,\ r\sin\theta\partial\lambda , \qquad (7\text{-}7)$$

in agreement with (7-3) and (7-4). This fully explains the terms in (7-5) containing partial derivatives; remaining terms containing u_r , u_θ , u_λ directly are due to the curvilinear character of the spherical coordinate system.

For the spherical components f_r, f_θ, f_λ of the elastic force vector

$$f = \operatorname{div} \underline{P}_1 \qquad (7\text{-}8)$$

we get

$$rf_r = r\frac{\partial p_{rr}}{\partial r} + \frac{\partial p_{r\theta}}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial p_{r\lambda}}{\partial \lambda} + 2p_{rr} - p_{\theta\theta} - p_{\lambda\lambda} + p_{r\theta}\cot\theta \ ,$$

$$rf_\theta = r\frac{\partial p_{r\theta}}{\partial r} + \frac{\partial p_{\theta\theta}}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial p_{\theta\lambda}}{\partial \lambda} + (p_{\theta\theta} - p_{\lambda\lambda})\cot\theta +$$

$$+ 3p_{r\theta} \ , \qquad (7\text{-}9)$$

$$rf_\lambda = r\frac{\partial p_{r\lambda}}{\partial r} + \frac{\partial p_{\theta\lambda}}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial p_{\lambda\lambda}}{\partial \lambda} + 3p_{r\lambda} + 2p_{\theta\lambda}\cot\theta \ .$$

Again the analogy to the rectangular case

$$f = \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{xy}}{\partial y} + \frac{\partial p_{xz}}{\partial z} \ , \ \text{etc.} \qquad (7\text{-}10)$$

helps to make the spherical equations transparent.

The volume dilatation (5-11) takes the form

$$\Theta = \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_\lambda}{\partial \lambda} + \frac{2}{r}u_r + \frac{\cot\theta}{r}u_\theta \ . \qquad (7\text{-}11)$$

Especially simple becomes the quantity (6-14) :

$$\begin{aligned} n &= \underline{u} \cdot \operatorname{grad} W \\ &= u_r \frac{\partial W}{\partial r} + u_\theta \frac{1}{r} \frac{\partial W}{\partial \theta} + u_\lambda \frac{1}{r \sin\theta} \frac{\partial W}{\partial \lambda} \; . \end{aligned} \qquad (7\text{-}12)$$

We identify W with the reference potential W_0 , which, in the spherical approximation, is spherically symmetric, depending only on r :

$$W = W(r) \quad . \qquad (7\text{-}13)$$

Then

$$\frac{\partial W}{\partial r} = W'(r) \qquad (7\text{-}14)$$

and the other partial derivatives are zero. Thus (7-12) reduces to

$$n = u_r W'(r) \; . \qquad (7\text{-}15)$$

We finally need the Laplacian operator

$$\Delta F = \frac{\partial^2 F}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 F}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 F}{\partial \lambda^2} + \frac{2}{r} \frac{\partial F}{\partial r} + \frac{\cot\theta}{r^2} \frac{\partial F}{\partial \theta} \; ; \qquad (7\text{-}16)$$

cf. (Heiskanen and Moritz, 1967, p. 19).

<u>Second-degree harmonic displacements.</u> In order to simplify and solve the system of partial differential equations (7-2), it is customary to take for the components of the displacement vector $\underline{u}$ the following special form:

$$u_r = H(r)S(\theta,\lambda) = HS ,$$

$$u_\theta = L(r)\,\frac{1}{r}\frac{\partial S}{\partial \theta} = Lr^{-1}S_\theta , \qquad (7\text{-}17)$$

$$u_\lambda = L(r)\,\frac{1}{r\sin\theta}\frac{\partial S}{\partial \lambda} = L(r\sin\theta)^{-1}S_\lambda ,$$

where S_θ and S_λ denote partial derivatives (in contrast to u_θ , u_λ which simply are components).

Here $H(r)$ and $L(r)$ are functions of the radius vector r , whereas $S(\theta,\lambda)$ denotes any Laplace surface harmonic of second degree. For the present purpose (tidal effects on the direction of the rotation axis) it is appropriate to consider $S(\theta,\lambda)$ a spherical harmonic of degree $n = 2$ and order $m = 1$, that is, of the form

$$\begin{aligned} S(\theta,\lambda) &= P_{11}(\cos\theta)(a\cos\lambda + b\sin\lambda) \\ &= aR_{11}(\theta,\lambda) + bS_{11}(\theta,\lambda) , \end{aligned} \qquad (7\text{-}18)$$

where a and b are arbitrary constants. For notations cf. (Heiskanen and Moritz, 1967, p. 29), and for a justification of the restriction to $n = 2$ cf. (Moritz, 1980a, sec. 55); note also that (7-18) is in agreement with the form (2-40) of the present

report for the tidal potential.

Being a Laplace harmonic for $n = 2$, the function $S(\theta,\lambda)$ satisfies the partial differential equation

$$S_{\theta\theta} + \frac{1}{\sin^2\theta} S_{\lambda\lambda} + \cot\theta S_\theta = -6S \tag{7-19}$$

(Heiskanen and Moritz, 1967, p. 20, eq. (1-44)).

We also note that Molodensky (1961) uses the notation $T(r)$ instead of $L(r)$; we have restricted the letter T to toroidal oscillations, so that our notation $L(r)$ is consistent with sec. 3.

With (7-17) let us now compute the volume dilation. We find

$$\frac{\partial u_r}{\partial r} = H'(r)S(\theta,\lambda) = H'S \quad ; \tag{7-20}$$

primes will always denote d/dr, so that $H' = dH/dr$. Similarly

$$\frac{\partial u_\theta}{\partial \theta} = Lr^{-1}S_{\theta\theta} \quad , \qquad \frac{\partial u_\lambda}{\partial \lambda} = L(r\sin\theta)^{-1}S_{\lambda\lambda} \quad . \tag{7-21}$$

Thus (7-11) becomes

$$\begin{aligned}\Theta &= H'S + 2r^{-1}HS + r^{-2}L(S_{\theta\theta} + \frac{1}{\sin^2\theta} S_{\lambda\lambda} + \cot\theta S_\theta) \\ &= (H' + 2r^{-1}H - 6r^{-2}L)S\end{aligned}$$

by (7-19). This may be written in the form

$$\Theta = XS \tag{7-22}$$

with

$$X = H' + 2r^{-1}H - 6r^{-2}L \quad . \tag{7-23}$$

In the same straightforward way we get from (7-5) the components of the stress tensor :

$$\begin{aligned}
p_{rr} &= NS \ , \\
p_{\theta\theta} &= QS + 2\mu r^{-2}LS_{\theta\theta} \ , \\
p_{\lambda\lambda} &= QS + 2\mu r^{-2}L(\sin^{-2}\theta S_{\lambda\lambda} + \cot\theta S_{\theta}) \ , \\
p_{r\theta} &= r^{-3}MS_{\theta} \ , \\
p_{r\lambda} &= r^{-3}M\sin^{-1}\theta S_{\lambda} \ , \\
p_{\theta\lambda} &= 2\mu r^{-2}L\sin^{-1}\theta(S_{\theta\lambda} - \cot\theta S_{\lambda})
\end{aligned} \tag{7-24}$$

with the abbreviations

$$M = \mu(r^2H + r^2L' - 2rL) \ , \tag{7-25}$$

$$N = \overline{\lambda}X + 2\mu H' \ , \tag{7-26}$$

$$Q = \overline{\lambda}X + 2\mu r^{-1}H \quad . \tag{7-27}$$

Note that N is proportional to the <u>normal tension</u> p_{rr}, and that M is proportional to the <u>tangential tensions</u> $p_{r\theta}$ and $p_{r\lambda}$ acting on a surface $r = \text{const.}$

In the same way we get for the components of the elastic force (7-9) :

$$f_r = (N' - 6r^{-4}M + 4\mu r^{-1}H' - 4\mu r^{-2}H + 12\mu r^{-3}L)S ,$$

$$f_\theta = (r^{-3}M' + r^{-1}Q - 10\mu r^{-3}L)S_\theta , \qquad (7\text{-}28)$$

$$f_\lambda = (r^{-3}M' + r^{-1}Q - 10\mu r^{-3}L)\sin^{-1}\theta S_\lambda .$$

Once more we remark that the subscripts r, θ, λ in p_{rr}, $p_{r\theta}$, ..., f_λ denote components, whereas in S_θ, $S_{\theta\lambda}$, ... they denote partial derivatives.

The quantity n is immediately obtained from (7-15) and (7-17):

$$n = HSW' . \qquad (7\text{-}29)$$

We finally try the representation

$$V_e + V_1 = R(r)S(\theta,\lambda) = RS . \qquad (7\text{-}30)$$

A first justification is provided by the fact that the external tidal potential V_e is indeed a spherical harmonic with $n = 2$, $m = 1$, and that the response V_1 to the disturbance V_e can be expected to be proportional to $S(\theta,\lambda)$ as well. The final justification lies in the fact that the substitution (7-30) works, as will be seen.

Now we can write (7-2) in spherical coordinates:

$$f_r + \rho \frac{\partial}{\partial r} (V_e + V_1 + \eta) - \rho \Theta W' = 0 ,$$

$$f_\theta + \rho r^{-1} \frac{\partial}{\partial \theta} (V_e + V_1 + \eta) = 0 , \qquad (7\text{-}31)$$

$$f_\lambda + \rho (r \sin\theta)^{-1} \frac{\partial}{\partial \lambda} (V_e + V_1 + \eta) = 0 ;$$

here we have used (7-3), (7-8), and (7-13). We substitute (7-28), (7-29), and (7-30), differentiating according to

$$\frac{\partial}{\partial r} (V_e + V_1 + \eta) = S \frac{d}{dr} (R + HW') =$$

$$= S(R' + H'W' + HW'') ,$$

$$\frac{\partial}{\partial \theta} (V_e + V_1 + \eta) = (R + HW') S_\theta .$$

The first two equations of (7-31) then give

$$N' - 6r^{-4}M + 4\mu r^{-1}H' - 4\mu r^{-2}H + 12\mu r^{-3}L +$$

$$+ \rho(R' + H'W' + HW'' - XW') = 0 , \qquad (7\text{-}32)$$

$$r^{-3}M' + r^{-1}Q - 10\mu r^{-3}L + r^{-1}\rho(R + HW') = 0 ; \qquad (7\text{-}33)$$

the third equation of (7-31) provides a check for (7-33).

This is the result of the elasticity equations (7-2). They are not yet sufficient to fully determine the problem. We still need Poisson's equation (6-30) ,

$$\Delta V_1 = -4\pi G\rho_1 =$$

$$= 4\pi G(\underline{u} \cdot \operatorname{grad}\rho + \rho\Theta)$$

by (6-24), writing ρ for ρ_0 as usual. For spherical symmetry,

$$\rho = \rho(r) , \tag{7-34}$$

this becomes

$$\Delta V_1 = 4\pi G(\rho' u_r + \rho\Theta) = 4\pi G(\rho' H + \rho X)S . \tag{7-35}$$

Since the tidal potential V_e is harmonic (outside of sun and moon) we have

$$\Delta V_e = 0 , \tag{7-36}$$

so that

$$\Delta(V_e + V_1) = \Delta(RS) = \Delta V_1 \tag{7-37}$$

as given by (7-35). From (7-16), (7-19), and (7-30) we obtain

$$\Delta(RS) = (R'' + 2r^{-1}R' - 6r^{-2}R)S . \tag{7-38}$$

The comparison of (7-35) and (7-38), using (7-37), finally gives

$$R'' + 2r^{-1}R' - 6r^{-2}R = 4\pi G(\rho' H + \rho X) . \qquad (7\text{-}39)$$

We now introduce a new auxiliary variable $P(r)$ by

$$P = r^2(R' - 4\pi G\rho H) \qquad (7\text{-}40)$$

(Molodensky denotes it by L). We solve (7-40) for R' , differentiate to obtain R'' and substitute into (7-39). The result is

$$P' = 6R + 4\pi G\rho r^2(X - H' - 2r^{-1}H) . \qquad (7\text{-}41)$$

The system of the two first-order differential equations (7-40) and(7-41) is equivalent to the second-order differential equation (7-39); the advantage is that now ρ' is eliminated, so that ρ need not be differentiated. This is important since the differentiation of an empirically (and not too well) determined quantity such as ρ is problematical.

Equations (7-25), (7-26), (7-32), (7-33), (7-40), and (7-41) will be our basic equations. After elimination of X and Q by (7-23) and (7-27) they become

$$M = \mu(r^2L' - 2rL + r^2H) ,$$

$$N = (\lambda + 2\mu)H' + \lambda(2r^{-1}H - 6r^{-2}L) ,$$

$$P = r^2(R' - 4\pi G\rho H) ,$$

(7-42)

$$P' = 6R - 24\pi G\rho L \quad , \tag{7-42}$$

$$N' - 6r^{-4}M + 4\mu r^{-1}H' - 4\mu r^{-2}H + 12\mu r^{-3}L +$$
$$+ \rho(R' + HW'' - 2r^{-1}HW' - 6r^{-2}LW') = 0 \; ,$$

$$r^{-3}M' + \lambda r^{-1}H' + 2(\lambda + \mu)r^{-2}H - (6\lambda + 10\mu)r^{-3}L +$$
$$+ \rho r^{-1}(R + HW') = 0 \; .$$

This is a system of 6 ordinary differential equations of the first order for the 6 unknown functions $H(r)$, $L(r)$, $M(r)$, $N(r)$, $P(r)$, $R(r)$. The density $\rho(r)$, the Lamé parameters $\lambda(r)$, $\mu(r)$ (we have again written λ instead of $\overline{\lambda}$) , and the potential $W(r)$ are considered known from a standard earth model.

A basic advantage of the system in its present form, which goes back to (Molodensky, 1953), is that it does not contain derivatives of the empirical quantities λ, μ, ρ .

The equation system (7-42) will be further considered in the next section.

8. SOLUTION FOR THE MANTLE

The basic equation system (7-42) may be solved algebraically for the derivatives H', L', ..., R' . The result has the form

$$
\begin{aligned}
H' &= a_{11}H + a_{12}L + a_{13}R + a_{14}M + a_{15}N + a_{16}P \,, \\
L' &= a_{21}H + a_{22}L + a_{23}R + a_{24}M + a_{25}N + a_{26}P \,, \\
R' &= a_{31}H + a_{32}L + a_{33}R + a_{34}M + a_{35}N + a_{36}P \,, \\
M' &= a_{41}H + a_{42}L + a_{43}R + a_{44}M + a_{45}N + a_{46}P \,, \\
N' &= a_{51}H + a_{52}L + a_{53}R + a_{54}M + a_{55}N + a_{56}P \,, \\
P' &= a_{61}H + a_{62}L + a_{63}R + a_{64}M + a_{65}N + a_{66}P \,.
\end{aligned}
\tag{8-1}
$$

The coefficients a_{ik} are functions of r . For instance, the comparison between the last equation of (8-1) and the fourth equation of (7-42) shows that

$$
a_{61} = 0 \,, \qquad a_{62} = -24\pi G\rho \,, \qquad a_{63} = 6 \,,
$$
$$
a_{64} = a_{65} = a_{66} = 0 \,.
$$

If λ, μ, ρ, W are given as functions of r , then the coefficients $a_{ik}(r)$ can be computed and the system of differential equations (8-1) integrated by a numerical integration technique, provided suitable boundary values

$$
H_0,\ L_0,\ R_0,\ M_0,\ N_0,\ P_0, \tag{8-2}
$$

are given. We take them to be the values of the corresponding quantities at the earth's surface, considered a sphere of radius

$a = 6371$ km :

$$H_0 = H(a), \; L_0 = L(a), \; \ldots\ldots, \; P_0 = P(a) \; . \qquad (8\text{-}3)$$

A slightly different approach is the following. We solve the system (8-1) six times, first with boundary values

$$H_0 = 1, \; L_0 = R_0 = \cdot \cdot \cdot = P_0 = 0 \; , \qquad (8\text{-}4)$$

denoting the solution by $H_1(r), L_1(r), \ldots, P_1(r)$. Then we solve with boundary values

$$H_0 = 0, \; L_0 = 1, \; R_0 = M_0 = N_0 = P_0 = 0 \; , \qquad (8\text{-}5)$$

obtaining solutions $H_2(r), L_2(r), \ldots\ldots, P_2(r)$. We continue with $R_0 = 1$, all other boundary values being zero, etc. Finally we take

$$H_0 = L_0 = \cdot \cdot \cdot \cdot = N_0 = 0 \; , \quad P_0 = 1 \; , \qquad (8\text{-}6)$$

denoting the solution by $H_0(r), L_6(r), \ldots., P_6(r)$.

Then the other solution of (8-1) for general boundary values (8-2) is given by

$$\begin{aligned} H(r) &= H_0H_1(r) + L_0H_2(r) + \cdot \cdot \cdot + P_0H_6(r) \; , \\ L(r) &= H_0L_1(r) + L_0L_2(r) + \cdot \cdot \cdot + P_0L_6(r) \; , \\ &\;\;\vdots \\ P(r) &= H_0P_1(r) + L_0P_2(r) + \cdot \cdot \cdot + P_0P_6(r) \; . \end{aligned} \qquad (8\text{-}7)$$

This linear dependence of the solution on the boundary values follows from the linearity of the differential equations (8-1): a linear combination of solutions is also a solution. The boundary conditions are also satisfied since, for $r = a$, (8-7) gives

$$H(a) = H_0 \cdot 1 + L_0 \cdot 0 + \cdot \cdot \cdot + P_0 \cdot 0 = H_0 \text{ , etc.}$$

These two approaches, the direct approach using the boundary values (8-2), and the indirect approach with (8-4), (8-5), ..., correspond to the solution of a system of algebraic linear equations, first by a direct method such as Cholesky's, and second by means of the inversion of the matrix.

The second approach is particularly transparent and simple in the present case, in view of the special form of the boundary values to be considered now.

The boundary values. First we note that two boundary values are even zero. This follows from the condition that the earth's surface is free from stress. The stress vector $\underline{p}$ acting on the sphere $r = a$ is given by (5-13), with

$$\underline{n} = (1,0,0)$$

for the radial unit vector in spherical coordinates (only the first, radial, component of $\underline{n}$ is different from zero). We get

$$\underline{p} = \begin{bmatrix} p_{rr} & p_{r\theta} & p_{r\lambda} \\ p_{r\theta} & p_{\theta\theta} & p_{\theta\lambda} \\ p_{r\lambda} & p_{\theta\lambda} & p_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p_{rr} \\ p_{r\theta} \\ p_{r\lambda} \end{bmatrix} \tag{8-8}$$

which must be zero at the earth's surface $r = a$. Thus all tensions p_{rr} (normal) and $p_{r\theta}$ and $p_{r\lambda}$ (tangential) vanish for $r = a$. By (7-24) this leads to the conditions

$$M_0 = M(a) = 0 \ , \tag{8-9a}$$

$$N_0 = N(a) = 0 \ . \tag{8-9b}$$

We further put

$$H_0 = H(a) = h \ , \tag{8-10}$$

$$L_0 = L(a) = 1 \ . \tag{8-11}$$

For the external potential V_e we have found the expression (2-43) :

$$V_e = \kappa r^2 S_1 \tag{8-12}$$

where

$$S_1 = S_1(\theta,\lambda,t) = \sin\theta\cos\theta\cos(\sigma t - \lambda) \ . \tag{8-13}$$

For $r = a$ this gives

$$V_e(a) = \kappa a^2 S_1 \ , \tag{8-14}$$

so that, by (8-12)

$$V_e(r) = \frac{r^2}{a^2} V_e(a) \ . \tag{8-15}$$

It is clear that (8-14) represents a spherical surface harmonic ($n = 2$, $m = 1$), and we may therefore define the function $S(\theta,\lambda)$ in (7-17) to be identical to $V_e(a)$:

$$S(\theta,\lambda) = V_e(a) = \kappa a^2 S_1 \quad . \tag{8-16}$$

Then (7-30) and (8-15) give

$$V_1(r) = R(r)S - V_e(r) = [R(r) - \frac{r^2}{a^2}] S \tag{8-17}$$

and

$$V_1(a) = [R(a) - 1]S = [R(a) - 1]V_e(a) \quad . \tag{8-18}$$

Denote by

$$k = R(a) - 1 \tag{8-19}$$

the factor of proportionality between the indirect effect V_1 and the tidal potential V_e at $r = a$. It is nothing else than the well-known <u>Love number</u> for the potential.

Thus,

$$R_0 = R(a) = 1 + k \tag{8-20}$$

expresses the boundary value of R in terms of the Love number k. Now we see that also h and l in (8-10) and (8-11) can be identified with the usual Love numbers. In fact, the definition of the Love numbers (cf. Moritz, 1980a, p. 480) reads

$$u_r = \frac{h}{g} V_e \ ,$$

$$u_\theta = \frac{1}{g} \frac{\partial V_e}{\partial \theta} \ , \tag{8-21}$$

$$u_\lambda = \frac{1}{g} \frac{1}{\sin\theta} \frac{\partial V_e}{\partial \lambda}$$

for $r = a$, where g is mean gravity at the earth's surface. Putting $r = a$ in (7-17) and comparing to (8-21) we find with $S = V_e$:

$$H(a) = g^{-1} h \ ,$$

$$L(a) = a g^{-1} l \ . \tag{8-22}$$

If we chose units such that

$$a = 1 \ , \quad g = 1 \ , \tag{8-23}$$

we get (8-10) and (8-11).

The last boundary value, P_0 , can be expressed in terms of the Love number k by

$$P_0 = P(a) = 2 - 3k \ . \tag{8-24}$$

The derivation of this simple relation is somewhat lengthy and will be deferred to the end of the present section.

Equations (8-9) through (8-11), as well as (8-20) and (8-24), give the boundary values. Thus the solution (8-7) may be written

$$H(r) = hH_1(r) + lH_2(r) + (1+k)H_3(r) + (2-3k)H_6(r) \ ,$$
$$L(r) = hL_1(r) + lL_2(r) + (1+k)L_3(r) + (2-3k)L_6(r) \ ,$$
$$\vdots \qquad\qquad (8\text{-}25)$$
$$P(r) = hP_1(r) + lP_2(r) + (1+k)P_3(r) + (2-3k)P_6(r) \ .$$

The boundary values on which the solution depends, are only the Love numbers h, k, l. These parameters h, k, l will be determined in the next section by means of the interaction between core and mantle.

The functions $H_4(r)$, $H_5(r)$, $L_4(r)$, $L_5(r)$, ..., $P_4(r)$, $P_5(r)$ do not enter in (8-25) and are not needed.

Derivation of (8-24). Consider the surface of the earth before and after deformation (Fig. 8.1). Consider further the

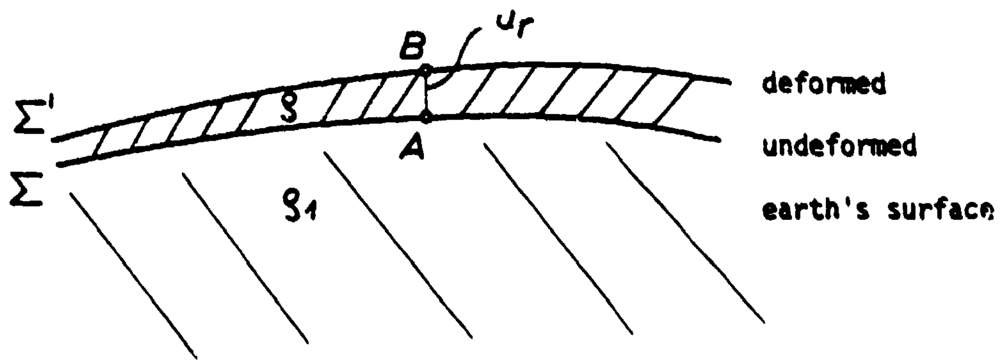

FIGURE 8.1. The earth's surface before and after deformation.

potential V_e. This potential may be regarded as being produced by the small density ρ_1 distributed inside the earth's surface, combined with a surface layer of density

$$\kappa = \rho u_r \ . \tag{8-26}$$

In fact, the space between the surfaces Σ and Σ', which was empty before the deformation, is filled with density ρ after the deformation; and the attraction between this thin layer between Σ and Σ' may be very well approximated by that of a material surface of density (8-26).

Now we know that the normal derivative $\partial/\partial n$ of the potential V of a surface layer undergoes a discontinuity at the surface

$$\left(\frac{\partial V}{\partial n}\right)_e - \left(\frac{\partial V}{\partial n}\right)_i = -4\pi G\kappa \ , \tag{8-27}$$

cf. (Heiskanen and Moritz, 1967, p. 6); the subscripts e and i denote exterior and interior derivatives, respectively. In our case, the normal derivative is the radial derivative $\partial/\partial r$, V is V_1 , and κ is given by (8-26). Thus,

$$\left(\frac{\partial V_1}{\partial r}\right)_e - \left(\frac{\partial V_1}{\partial r}\right)_i = -4\pi G\rho u_r \ . \tag{8-28}$$

By (8-17) we have

$$V_1(r) = [R(r) - \frac{r^2}{a^2}]\, S \qquad \text{for} \quad r < a \ , \tag{8-29}$$

that is, inside the earth. Differentiating and putting $r = a$ we get

$$\left(\frac{\partial V_1}{\partial r}\right)_i = [R'(a) - 2a^{-1}]S \quad . \tag{8-30}$$

Outside the earth, the potential V_1 is harmonic as any gravitational potential is. Thus it is given by the expression

$$V_1(r) = [\, R(a) - 1 \,]\frac{a^3}{r^3}\, S(\theta,\lambda) \qquad \text{for} \quad r > a \ . \tag{8-31}$$

In fact, $S(\theta,\lambda)$ is a spherical surface harmonic of degree $n = 2$, so that the dependence on r must necessarily have the form

$$\frac{1}{r^{n+1}} = \frac{1}{r^3}$$

for a harmonic function everywhere regular outside a sphere; cf. (Heiskanen and Moritz, 1967), pp. 20 and 34. Furthermore, for $r = a$ it must reduce to (8-18). These two conditions uniquely determine (8-31).

Differentiating (8-31) and putting $r = a$ we get

$$\left(\frac{\partial V_1}{\partial r}\right)_e = -3a^{-1}[R(a) - 1]S \ . \tag{8-32}$$

Now we substitute (8-30) and (8-32), together with

$$(u_r)_{r=a} = H(a)S \ , \tag{8-33}$$

into (8-28) :

$$-3a^{-1}R(a) + 3a^{-1} - R'(a) + 2a^{-1} = -4\pi G\rho H(a)$$

or

$$R'(a) - 4\pi G\rho H(a) = a^{-1}[5 - 3R(a)] .$$

By (7-40), the left-hand side is $a^{-2}P(a)$, so that

$$P(a) = a[5 - 3R(a)] .$$

Substituting (8-20) we find

$$P(a) = a(2 - 3k) , \tag{8-34}$$

and with (8-23) we get (8-24), which was to be derived.

9. CORE - MANTLE BOUNDARY CONDITIONS

As in the last section, we put

$$a = 1 \tag{9-1}$$

for the radius of the earth's surface, approximated by a sphere. To the same approximation, the core-mantle boundary will be a sphere of radius

$$b = 0.55 \ . \tag{9-2}$$

We shall now derive various conditions to be satisfied on the surface $r = b$.

Parameter M : no tangential tension. The components of the stress vector (8-8) must be continuous across the core-mantle boundary $r = b$. Since the core is an incompressible fluid, there is no tangential tension there. By continuity, the tangential tensions $p_{r\theta}$ and $p_{r\lambda}$ must be zero also on $r = b$, which by (7-24) leads to the condition

$$M(b) = 0 \ . \tag{9-3}$$

By (8-25) this may be written

$$hM_1(b) + lM_2(b) + (1+k)M_3(b) + (2-3k)M_6(b) = 0 \ . \tag{9-4}$$

Now the $M_i(b)$ are constants which are known if the functions $M_i(r)$ have been computed in the way described in sec. 8. Therefore, (9-4) gives a condition to be satisfied for the Love numbers

h, k, l . Being independent of the core structure, it is the simplest boundary condition. For example, Molodensky (1961) gives

$$0.7053\,k - 0.4793\,h - 3.7155\,l + 0.3833 = 0 \qquad (9\text{-}5)$$

for his Model 2 (cf. sec. 10); numerical values will, of course, be slightly different.

Parameter H : continuity of normal displacement. The normal displacement q of the core-mantle boundary is given by (7-17) :

$$q = u_r(b) = H(b)S(\theta,\lambda) \quad . \qquad (9\text{-}6)$$

By (7-15) we have on $r = b$

$$\eta = qW'(b) = H(b)W'(b)S(\theta,\lambda) \; . \qquad (9\text{-}7)$$

Eq. (8-16) gives with (9-1)

$$S(\theta,\lambda) = \kappa S_1 \qquad (9\text{-}8)$$

where

$$S_1 = \sin\theta\cos\theta\cos(\sigma t - \lambda) \; . \qquad (9\text{-}9)$$

In the liquid core, η has the form (2-47), which on the boundary $r = b$ becomes

$$\eta = -Cb^2 S_1 \; . \qquad (9\text{-}10)$$

The comparison between (9-7) and (9-10), taking account of (9-8), shows that

$$C = -\kappa b^{-2}H(b)W'(b) \ . \tag{9-11}$$

The boundary condition expresses the parameter C in (2-47) in terms of $H(b)$ and, by (8-25) for $r = b$, finally in terms of the Love numbers h, k, l.

Before going to the next condition, we point out that the spherical approximation, used from sec. 7 on, is legitimate for the mantle including its lower boundary $r = b$. In the core, however, the ellipticity plays an essential role and must be taken into account. This requires some care.

<u>Parameter R : continuity of potential.</u> In the core we have by (2-39) with (2-19) and (2-46) :

$$\begin{aligned} V &= V_e + V_1 + \left(1 + \frac{\sigma}{\Omega}\right)\phi \\ &= (A - B)(xz\cos\sigma t + yz\sin\sigma t) \\ &= (A - B)r^2 S_1 \ . \end{aligned} \tag{9-12a}$$

In the mantle (7-30) holds:

$$V_e + V_1 = RS \ . \tag{9-12b}$$

Since the potential is continuous across the core boundary, both expressions must be equal for $r = b$:

$$(A - B)b^2 S_1 = R(b)\kappa S_1 \ . \tag{9-13}$$

Here we have used (9-8) and neglected the term

$$\left(1 + \frac{\sigma}{\Omega}\right)\phi = \frac{\Omega + \sigma}{\Omega} \quad . \tag{9-14}$$

This is permissible since $\sigma \doteq -\Omega$ for the main tidal frequencies and since the quantity ϕ is very small. Thus (9-13) gives

$$A - B = \kappa b^{-2} R(b) \ . \tag{9-15}$$

This equation relates the core parameters A and B to $R(b)$ and hence, by (8-25) to h, k, l .

Parameter N : equality of normal stress. Above we have pointed out the continuity of the stress vector across the boundary $r = b$ and used the fact to derive the condition (9-3). Now we take the normal component p_{rr} . In the core this corresponds to $-p_1$ where p_1 is the anomalous pressure (2-32). Hence p_{rr} for the mantle and $-p_1$ for the core must be equal on the boundary $r = b$, which leads to the condition

$$p_{rr} + p_1 = 0 \ . \tag{9-16}$$

Now p_{rr} is given by (7-24) :

$$p_{rr} = NS = N\kappa S_1 \tag{9-17}$$

and p_1 may be found from (2-48) :

$$p_1 = \rho(A - C)r^2 S_1 \quad . \tag{9-18}$$

We put $r = b$ and substitute in (9-16), obtaining

$$\kappa N(b) + \rho b^2(A - C) = 0 . \qquad (9\text{-}19)$$

Here we may eliminate C by (9-11) and express A in terms of B by (9-15). The result is

$$\rho_{core}^{-1} N(b) + R(b) + H(b)W'(b) + \kappa^{-1} b^2 B = 0 , \qquad (9\text{-}20)$$

writing $\rho = \rho_{core}$ for the (constant) density of the core.

<u>Parameter P : discontinuity of the potential gradient.</u>
On the earth's surface $r = a = 1$ we have the condition (8-28) :

$$\left(\frac{\partial V_1}{\partial r}\right)_e - \left(\frac{\partial V_1}{\partial r}\right)_i = -4\pi G \rho u_r , \qquad (9\text{-}21)$$

expressing the discontinuity of $\partial V_1/\partial r$ across this surface. This is due to the density jumping from ρ inside the surface to the value zero outside, so that ρ in (9-21) represents this density jump.

Across the core-mantle boundary $r = b$, the density jumps from ρ_{core} below to ρ_{mantle} above. Thus (9-21) takes for the surface $r = b$ the form

$$\left(\frac{\partial V_1}{\partial r}\right)_{mantle} - \left(\frac{\partial V_1}{\partial r}\right)_{core} = -4\pi G(\rho_{core} - \rho_{mantle}) u_r , \qquad (9\text{-}22)$$

replacing ρ by the density jump $\rho_{core} - \rho_{mantle}$; the notation

$$\left(\frac{\partial V_1}{\partial r}\right)_e = \left(\frac{\partial V_1}{\partial r}\right)_{\text{mantle}}$$

is clear since the external derivative is found by approaching the surface $r = b$ from above, that is, through the mantle, and similarly for $(\partial V/\partial r)_i$.

We may write (9-22) in the form

$$\left(\frac{\partial V_1}{\partial r}\right)_{\text{mantle}} - 4\pi G\rho_{\text{mantle}} u_r = \left(\frac{\partial V_1}{\partial r}\right)_{\text{core}} - 4\pi G\rho_{\text{core}} u_r, \qquad (9\text{-}23)$$

which shows that the quantity

$$\frac{\partial V_1}{\partial r} - 4\pi G\rho u_r \qquad (9\text{-}24)$$

is continuous across the boundary $r = b$.

The external tidal potential V_e is harmonic inside the earth and therefore continuous, together with its derivative $\partial V_e/\partial r$, across the surface $r = b$. The same holds for the incremental centrifugal potential ϕ defined by (1-31). Putting

$$V = V_e + V_1 + \left(1 + \frac{\sigma}{\Omega}\right)\phi$$

as in (9-12a) we therefore conclude that the quantity

$$\frac{\partial V}{\partial r} - 4\pi G\rho u_r \qquad (9\text{-}25)$$

is also continuous across the boundary $r = b$.

In the mantle we may neglect ϕ and replace V by (9-12b),

so that

$$\frac{\partial V}{\partial r} = R'S \ ,$$

$$\frac{\partial V}{\partial r} - 4\pi G\rho u_r = (R' - 4\pi G\rho H)S = Pr^{-2}S \ ; \tag{9-26}$$

we have used (7-17) and (7-40).

In the core, V is given by (9-12a), so that

$$\frac{\partial V}{\partial r} = (A - B)2rS_1 \ , \tag{9-27}$$

$$-4\pi G\rho u_r = -3gr^{-1}u_r \ , \tag{9-28}$$

introducing gravity g by (2-61). Thus

$$\frac{\partial V}{\partial r} - 4\pi G\rho u_r = (A - B)2rS_1 - 3gr^{-1}u_r \ . \tag{9-29}$$

In view of the continuity of (9-25), the expressions (9-26) and (9-29) must be equal at the boundary $r = b$:

$$P(b)b^{-2}S = (A - B)2bS_1 + 3W'(b)b^{-1}H(b)S \ . \tag{9-30}$$

Here we have put $g(b) = -W'(b)$, so that, by (2-61)

$$W'(b) = -\frac{4\pi G}{3} b\rho_{core} \ . \tag{9-31}$$

Finally we take into account (9-15) and (9-8), obtaining

$$P(b) = 2bR(b) + 3bW'(b)H(b) \ . \tag{9-32}$$

<u>Equations for Love numbers and fluid parameters.</u> The basic equations are the boundary conditions (9-3), (9-11), (9-15), (9-20), and (9-32). We collect them now, replacing the fluid constants A,B,C by the equivalent constants α, β, γ related to them by (2-20), (2-64), and (2-66); this is preferable since the symbols A and B will be used for the principal moments of inertia and since α, β, γ are dimensionless. Thus we get, also using (9-31),

$$M(b) = 0 \ ,$$

$$e^2 b \kappa^{-1} \gamma - H(b) = 0 \ ,$$

$$e^2 b^{-1} W'(b) \kappa^{-1} \alpha - \sigma(\sigma + 2\Omega)\kappa^{-1}\beta - b^{-2} R(b) = 0 \ , \qquad (9\text{-}33)$$

$$\rho_{core}^{-1} N(b) + R(b) + H(b)W'(b) + b^2 \sigma(\sigma + 2\Omega)\kappa^{-1}\beta = 0 \ ,$$

$$2bR(b) - P(b) + 3bW'(b)H(b) = 0 \ .$$

We combine these formulas with an equation following from (2-58) and (2-65) :

$$\frac{1}{2}(\gamma - \beta)e^2 + \frac{\sigma + \Omega}{\sigma}\beta - \frac{\Omega}{\sigma}\varepsilon = 0 \ . \qquad (9\text{-}34)$$

The quantities $H(b)$, $M(b)$, $P(b)$, $R(b)$ are linear combinations of h, k, l with known coefficients $H_1(b)$, $H_2(b)$, etc., cf. (8-25); the constant $W'(b)$, given by (9-31), is likewise known. Therefore, (9-33) and (9-34) represent a system of 6 equations for the 7 unknowns h, l, α, β, γ, ε.

Therefore, if we could find an independent seventh relation, then we could compute these 7 unknowns, getting a unique solution for any desired value of the frequency σ . Such a relation is provided by the Euler-Liouville condition to be discussed in the next section.

10. THE EULER - LIOUVILLE CONDITION

A rigid or nonrigid body rotating with an angular velocity vector $\underline{\omega}$ satisfies the equation

$$\frac{\partial \underline{H}}{\partial t} + \underline{\omega} \times \underline{H} = \underline{L} , \tag{10-1}$$

where $\underline{H}$ is the angular momentum, and $\underline{L}$ represents the torque (Moritz, 1980b, p. 13); this is Euler's equation. A simple consequence is Liouville's equatior

$$\frac{\partial}{\partial t} (\underline{C}\,\underline{\omega} + \underline{h}) + \underline{\omega} \times (\underline{C}\,\underline{\omega} + \underline{h}) = \underline{L} \tag{10-2}$$

where $\underline{C}$ is the inertia tensor (ibid., p. 14), and $\underline{h}$ is the relative angular momentum.

We put

$$\underline{\omega} = \underline{\omega}_0 + \delta\underline{\omega} , \qquad \underline{C} = \underline{C}_0 + \underline{c} , \tag{10-3}$$

$$\omega_0 = \begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix} , \qquad \delta\underline{\omega} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} \Omega , \tag{10-4}$$

$$\underline{C}_0 = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{bmatrix} , \qquad \underline{c} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} \tag{10-5}$$

and linearize, disregarding squares and higher powers of the small quantities $\delta\underline{\omega}$ and $\underline{c}$. The result is

$$
\begin{aligned}
A\Omega\dot{m}_1 + (C-A)\Omega^2 m_2 + \Omega\dot{c}_{13} - \Omega^2 c_{23} + \dot{h}_1 - \Omega h_2 &= L_1 , \\
A\Omega\dot{m}_2 - (C-A)\Omega^2 m_1 + \Omega\dot{c}_{23} + \Omega^2 c_{13} + \dot{h}_2 + \Omega h_1 &= L_2 .
\end{aligned}
\tag{10-6}
$$

This is a slight and straightforward generalization of eq. (3-14) of (Moritz, 1980b, p. 17) due to the inclusion of the relative angular momentum, which is necessary since it is not convenient here to use Tisserand axes for the entire earth; cf. also (Munk and Macdonald, 1960, p. 37).

The relative angular momentum is given by

$$
\underline{h} = \iiint\limits_{\text{earth}} \underline{x} \times \dot{\underline{u}}\, dM = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} . \tag{10-7}
$$

This is eq. (3-4) of (Moritz, 1980b, p. 13), but with $\underline{u}$ now replaced by $\dot{\underline{u}}$, which is necessary since the relative velocity was denoted by $\underline{u}$ in the earlier report, but now $\underline{u}$ designates the displacement so that the relative velocity is $\dot{\underline{u}}$; the symbol dM denotes the mass element.

With $\underline{u} = (u,v,w)$ we have

$$
\underline{x} \times \dot{\underline{u}} = \begin{bmatrix} y\dot{w} - z\dot{v} \\ z\dot{u} - x\dot{w} \\ x\dot{v} - y\dot{u} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} , \tag{10-8}
$$

so that

$$h_i = \iiint\limits_{earth} f_i \, dM \ . \qquad (10\text{-}9)$$

We split up this integral into an integral over the core and an integral over the mantle:

$$h_i = h_i^c + h_i^m \qquad (10\text{-}10)$$

where

$$h_i^c = \iiint\limits_{core} f_i \, dM \ , \qquad (10\text{-}11)$$

$$h_i^m = \iiint\limits_{mantle} f_i \, dM \ . \qquad (10\text{-}12)$$

Let us now choose the coordinate axis to be Tisserand axes for the mantle. Tisserand axes are defined by the condition that the relative angular momentum $\underline{h}$ is zero (Moritz, 1980b, p. 15). For Tisserand axes for the mantle, the relative angular momentum due to the mantle only, is zero :

$$h_i^m = 0 \ . \qquad (10\text{-}13)$$

Thus

$$h_i = h_i^c = \iiint_{core} f_i \, dM \; ; \tag{10-14}$$

the relative angular momentum is produced by the core only.

The substitution of (3-12) into (10-8) gives immediately

$$f_1 = -\sigma(\nu - \beta)xy\sin\sigma t + \sigma(\nu - \beta)y^2\cos\sigma t - \sigma\beta z^2\cos\sigma t \tag{10-15}$$

for the liquid core. We substitute into (10-14) and perform the integration over the core. We integrate over the unperturbed state, for which the core is an ellipsoid of revolution, the z-axis coinciding with the axis of symmetry. Then

$$\iiint_{core} xy\,dM = 0 \;, \tag{10-16}$$

this product of inertia being zero because the z-axis is a principal axis of inertia (Heiskanen and Moritz, 1967, p. 62). We further have

$$C_c = \iiint_{core} (x^2 + y^2)\,dM = 2\iiint_{core} y^2\,dM \tag{10-17}$$

because of rotational symmetry; C_c is the polar principal moment of inertia of the core. For the equatorial principal moment of inertia of the core we have

$$A_c = \iiint_{core} (y^2 + z^2)\,dM = \frac{1}{2}C_c + \iiint_{core} z^2\,dM \; . \tag{10-18}$$

Hence

$$\iiint\limits_{\text{core}} y^2 dM = \frac{1}{2} C_c \ ,$$
$$\iiint\limits_{\text{core}} z^2 dM = A_c - \frac{1}{2} C_c \ . \tag{10-19}$$

Thus the substitution of (10-15) into (10-14) and integration gives, in view of (10-19),

$$h_1 = [\frac{1}{2} \sigma(\nu - \beta) C_c - \sigma\beta(A_c - \frac{1}{2} C_c)]\cos\sigma t \tag{10-20}$$

or, after some algebra using (2-58) ,

$$h_1 = [\Omega(\beta - \varepsilon)A_c + \frac{1}{2} \sigma\nu(C_c - A_c)]\cos\sigma t \ ,$$
$$h_2 = [\Omega(\beta - \varepsilon)A_c + \frac{1}{2} \sigma\nu(C_c - A_c)]\sin\sigma t \ . \tag{10-21}$$

We have added the analogous expression for h_2 , which can be obtained in the same way.

Now we have to determine the (negative) products of inertia c_{13} and c_{23} according to (10-5); by definition (Moritz, 1980b, p. 9) there is

$$c_{13} = -D_{xz} = -\iiint xzdM \ , \quad c_{23} = -D_{yz} = -\iiint yzdM \ . \tag{10-22}$$

We shall calculate them separately for tidal deformation and for rotational deformation.

Tidal deformation. Consider the spherical-harmonic development of the external gravitational potential of the (deformed) earth:

$$V(r,\theta,\lambda) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{P_{nm}(\cos\theta)}{r^{n+1}} (A_{nm}\cos m\lambda + B_{nm}\sin m\lambda) . \quad (10\text{-}23)$$

By (Heiskanen and Moritz, 1967, p. 61, eq. (2-44c)) we have

$$A_{21} = GD_{xz} = -Gc_{13} , \qquad B_{21} = GD_{yz} = -Gc_{23} , \qquad (10\text{-}24)$$

denoting the gravitational constant by G .

The gravitational potential of the undeformed earth, considered an ellipsoid of revolution, has $D_{xz} = 0 = D_{yz}$. Thus the products of inertia are solely due to the deformation, that is, to the potential V_1 , which is the difference of the gravitational potentials of the deformed and the undeformed earth. For $r = a = 1$ we have by (8-18), (8-19), (9-8), and (9-9)

$$V_1 = kV_e = kS = k\kappa\sin\theta\cos\theta\cos(\sigma t - \lambda) \qquad (10\text{-}25)$$

or

$$V_1 = \frac{1}{3} k\kappa P_{21}(\cos\theta)(\cos\sigma t\cos\lambda + \sin\sigma t\sin\lambda) . \qquad (10\text{-}26)$$

We put $r = 1$ in (10-23) and compare the terms with $n = 2$, $m = 1$ with (10-26). This gives immediately

$$A_{21} = \frac{1}{3} k\kappa\cos\sigma t , \qquad B_{21} = \frac{1}{3} k\kappa\sin\sigma t , \qquad (10\text{-}27)$$

so that, by (10-24)

$$c_{13} = -\frac{1}{3} G^{-1}\kappa k\cos\sigma t , \qquad c_{23} = -\frac{1}{3} G^{-1}\kappa k\sin\sigma t , \qquad (10\text{-}28)$$

which determines c_{13} and c_{23} for the tidal deformation.

Rotational deformation. This is due to the deformation of the earth by the centrifugal force, similar to the distorting

force acting on a rotating unbalanced wheel (Moritz, 1980b, p. 2⊥). The incremental centrifugal potential by (1-31) is

$$\begin{aligned}\phi &= -\Omega^2\varepsilon(xz\cos\sigma t + yz\sin\sigma t)\\ &= -\Omega^2\varepsilon r^2\sin\theta\cos\theta\cos(\sigma t - \lambda) \quad .\end{aligned}$$

On the earth's surface, for $r = a$, this becomes

$$\phi = -\Omega^2\varepsilon a^2\sin\theta\cos\theta\cos(\sigma t - \lambda) \quad . \tag{10-29}$$

The comparison with (8-13) and (8-14) shows that ϕ acts like some kind of fictitious external potential with

$$\kappa = -\Omega^2\varepsilon \quad ; \tag{10-30}$$

this is not surprising since inertial forces such as the centrifugal force behave, in certain respects, like genuine forces. Thus the products of inertia due to rotational deformation are obtained by substituting (10-30) in (10-28) :

$$c_{13}^{RD} = \frac{1}{3}\, G^{-1}k\Omega^2\varepsilon\cos\sigma t \quad , \qquad c_{23}^{RD} = \frac{1}{3}\, G^{-1}k\Omega^2\varepsilon\sin\sigma t \quad . \tag{10-31}$$

The total products of inertia needed in (10-6) are the sum of (10-28) for tidal deformation (they should be written c_{13}^{TD} and c_{23}^{TD}) and (10-31) for rotational deformation:

$$\begin{aligned}c_{13} &= c_{13}^{TD} + c_{13}^{RD} = -\frac{1}{3}\, G^{-1}k(\kappa - \Omega^2\varepsilon)\cos\sigma t \quad ,\\ c_{23} &= c_{23}^{TD} + c_{23}^{RD} = -\frac{1}{3}\, G^{-1}k(\kappa - \Omega^2\varepsilon)\sin\sigma t \quad .\end{aligned} \tag{10-32}$$

Torque components L_1 and L_2. By (10-25) and (10-26) we write the tidal potential in the form

$$V_e = \frac{1}{3}\kappa P_{21}(\cos\theta)(\cos\sigma t\cos\lambda + \sin\sigma t\sin\lambda) \ . \qquad (10\text{-}33)$$

This represents v_{21} in (Moritz, 1980b, p. 3). By (ibid., p. 30) we therefore have

$$\begin{aligned} L_1 &= 3MJ_2 \frac{1}{P_{21}(\cos\theta)} V_e(\lambda = 90^\circ) \ , \\ L_2 &= -3MJ_2 \frac{1}{P_{21}(\cos\theta)} V_e(\lambda = 0^\circ) \ . \end{aligned} \qquad (10\text{-}34)$$

We put $\lambda = 90^\circ$ and $\lambda = 0^\circ$, respectively, in (10-33) and substitute in (10-34). The result is

$$\begin{aligned} L_1 &= \kappa MJ_2 \sin\sigma t \ , \\ L_2 &= -\kappa MJ_2 \cos\sigma t \ . \end{aligned} \qquad (10\text{-}35)$$

Euler-Liouville condition. Now we substitute (10-21), (10-32), and (10-35), together with (2-5), into (10-6). We get

$$\Big\{\Omega^2 C\varepsilon - (\sigma+\Omega)[\Omega A\varepsilon + \Omega A_c(\beta-\varepsilon) + \frac{1}{2}\sigma(C_c - A_c)\nu + \frac{1}{3}\Omega^3 G^{-1} k\varepsilon - \frac{1}{3}\Omega G^{-1}\kappa k]\Big\} \begin{Bmatrix} \sin\sigma t \\ -\cos\sigma t \end{Bmatrix} = \kappa MJ_2 \begin{Bmatrix} \sin\sigma t \\ -\cos\sigma t \end{Bmatrix} \ . \qquad (10\text{-}36)$$

By (Heiskanen and Moritz, 1967, p. 63) we have (there is $B = A$ because of rotational symmetry):

$$J_2 = J_{20} = \frac{C - A}{Ma^2} \quad ,$$

whence

$$\kappa M J_2 = \kappa(C - A) \tag{10-37}$$

since $a = 1$. Thus (10-36) gives

$$\Omega^2 C\varepsilon - (\sigma + \Omega)[\Omega A\varepsilon + \Omega A_c(\beta - \varepsilon) + \frac{1}{2}\sigma(C_c - A_c)\nu +$$
$$+ \frac{1}{3}\Omega^3 G^{-1} k\varepsilon - \frac{1}{3}\Omega\, G^{-1}\kappa k] = \kappa(C - A) \quad .$$

After some straightforward algebra, we finally find

$$\varepsilon - \frac{\sigma + \Omega}{\Omega}\left(\frac{A - A_c}{C}\,\varepsilon + \frac{A_c}{C}\,\beta + \frac{\sigma}{2\Omega}\,\frac{C_c - A_c}{C}\,\nu + \frac{\Omega^2 k}{3GC}\,\varepsilon -\right.$$
$$\left. - \frac{\kappa}{3GC}\,k\right) = \kappa\,\frac{C - A}{C\Omega^2} \quad . \tag{10-38}$$

This equation relates the polar motion parameter ε , the core parameter β , and the Love number k to the parameter κ , characterizing the external (tidal) potential by (10-33); note that ν is expressed in terms of β and ε by (2-58) :

$$\nu = 2\left(\frac{\sigma + \Omega}{\sigma}\,\beta - \frac{\Omega}{\sigma}\,\varepsilon\right) \quad . \tag{10-39}$$

It is a consequence of Euler's and Liouville's equations and will, therefore, be called <u>Euler-Liouville condition</u>.

It is sometimes convenient to introduce the so-called "secular Love number" k_s by

$$k_s = \frac{3G(C-A)}{a^5\Omega^2} \doteq 0.96 \text{ (dimensionless)} \tag{10-40}$$

(Munk and Macdonald, 1960, p. 26; Moritz, 1980b, p. 21). Then, with $a = 1$, eq. (10-38) takes the form

$$\varepsilon - \frac{\sigma+\Omega}{\Omega}\left(\frac{A-A_c}{C}\varepsilon + \frac{A_c}{C}\beta + \frac{\sigma}{2\Omega}\frac{C_c-A_c}{C}\nu + \right.$$
$$\left. + \frac{C-A}{C}\frac{k}{k_s}\varepsilon\right) = \kappa\frac{C-A}{C\Omega^2}\left(1 - \frac{k}{k_s}\frac{\sigma+\Omega}{\Omega}\right) . \tag{10-41}$$

For tidal frequencies we have $\sigma \doteq -\Omega$ so that $\sigma + \Omega$ will be small; so is $(C-A)C$. If we neglect the product of two small quantities, then the last term between parentheses on the left-hand side may be omitted; whence

$$\varepsilon - \frac{\sigma+\Omega}{\Omega}\left(\frac{A-A_c}{C}\varepsilon + \frac{A_c}{C}\beta + \frac{\sigma}{2\Omega}\frac{C_c-A_c}{C}\nu\right) =$$
$$= \kappa\frac{C-A}{C\Omega^2}\left(1 - \frac{k}{k_s}\frac{\sigma+\Omega}{\Omega}\right) . \tag{10-42}$$

This equation is identical to Molodensky's (1961) equation (67).

Equation (10-38), or its approximate form (10-42), represents the seventh condition needed to uniquely determine the 7 parameters $h, k, l, \alpha, \beta, \gamma, \varepsilon$ as mentioned at the end of sec. 9.

Determination of parameters. The relevant equations are thus (9-33), (9-34), and (10-38). We can solve them as follows. We first take from the system (9-33) the equations

$$
\begin{aligned}
&M(b) = 0 \, , \\
&\rho_{core}^{-1} N(b) + R(b) + H(b)W'(b) + b^2\sigma(\sigma+2\Omega)\kappa^{-1}\beta = 0 \, , \\
&2bR(b) - P(b) + 3bW'(b)H(b) = 0 \, .
\end{aligned}
\qquad (10\text{-}43)
$$

Since by (8-25) the quantities $H(b)$, $M(b)$, $N(b)$, $P(b)$, and $R(b)$ are linear combinations of the Love numbers h, k, l with constant coefficients, the system (10-43) has the form

$$
\begin{aligned}
&a_1 h + a_2 k + a_3 l = a_0 \, , \\
&b_1 h + b_2 k + b_3 l = b_0 + b_4\sigma(\sigma+2\Omega)\beta \, , \\
&c_1 h + c_2 k + c_3 l = c_0 \, .
\end{aligned}
\qquad (10\text{-}44)
$$

We can solve this equation for h, k, l, giving them as linear functions of $\sigma(\sigma+2\Omega)\beta$.

Molodensky (1961) has considered two models, both more realistic than our simple earth model featuring a homogeneous liquid core and an elastic mantle. Molodensky's two models have a more detailed core structure: Model 1 has a liquid core whose density

increases towards the center, and Model 2 has, in addition, a solid (elastic) inner core. Still, Molodensky's Model 1 is sufficiently similar to our simple model, so that his numerical values can be used as an illustration of the present formulas. Molodensky (1961) gives for Model 1 :

$$
\begin{aligned}
h &= 0.6206 + 0.4711 \times 10^{-3} \, \frac{\sigma(\sigma + 2\Omega)}{\Omega^2} \, \beta \quad , \\
k &= 0.3070 + 0.2384 \times 10^{-3} \, \frac{\sigma(\sigma + 2\Omega)}{\Omega^2} \, \beta \quad , \\
l &= 0.0904 - 0.0112 \times 10^{-3} \, \frac{\sigma(\sigma + 2\Omega)}{\Omega^2} \, \beta \quad .
\end{aligned}
\tag{10-45}
$$

These formulas show clearly that, <u>for a liquid core model, the Love numbers depend on frequency</u> (the strong dependence of β on the frequency σ is shown in the table below).

Now we can use the second equation of (9-33) together with (9-34), (10-38), and (10-39). We have

$$
\begin{aligned}
& e^2 b \kappa^{-1} \gamma - H(b) = 0 \ , \\
& \frac{1}{2} (\gamma - \beta) e^2 + \frac{\sigma + \Omega}{\sigma} \beta - \frac{\Omega}{\sigma} \varepsilon = 0 \ , \\
& \varepsilon - \frac{\sigma + \Omega}{\Omega} \left[\frac{A - C_c}{C} \varepsilon + \left(\frac{C_c - A_c}{C} \frac{\sigma}{\Omega} + \frac{C_c}{C} \right) \beta + \frac{\Omega^2 k}{3GC} \varepsilon - \right. \\
& \qquad \left. - \frac{\kappa}{3GC} k \right] = \kappa \, \frac{C - A}{C \, \Omega^2} \ ;
\end{aligned}
\tag{10-46}
$$

the last equation is obtained by eliminating ν from (10-38)

by means of (10-39). Since $H(b)$ is a linear combination of h, k, l and since h, k, l are functions of β by (10-45), (10-46) represents a system of three equations for $\beta, \gamma, \varepsilon$. If it has been solved, we finally get α from the third equation of (9-23).

For Model 1 with a surface flattening of 1/298.3 and the core values

$$e^2 = 0.005092 \,, \qquad C_c/C = 0.1180 \,. \tag{10-47}$$

Molodensky gives the following expressions:

$$\beta = \frac{41.87}{0.2136 - 100(\sigma + \Omega)/\sigma} + 1.9 \,, \tag{10-48}$$

$$\frac{\varepsilon}{\varepsilon_0} = 1 + \frac{C_c q}{C - A} (\beta - 4.3) \frac{\sigma + \Omega}{\Omega} \,. \tag{10-49}$$

Eq. (10-49) is to be understood in the following way. Denote by ε_0 the value of ε for a rigid earth (solid throughout). Then $k = 0$ because of rigidity, and $A_c = 0 = C_c$ because the core is missing. Thus, for a rigid earth, (10-38) reduces to

$$\left(1 - \frac{\sigma + \Omega}{\Omega} \frac{A}{C}\right) \varepsilon_0 = \kappa \frac{C - A}{C\Omega^2} \,. \tag{10-50}$$

This equation determines ε_0. The quantity q in (10-49) has the definition

$$q = \frac{\Omega^2 a}{g} = \frac{\text{centrifugal force at equator}}{\text{gravity at equator}} , \tag{10-51}$$

it reduces to Ω^2 for units in which $a = 1$, $g = 1$.

Thus $\varepsilon/\varepsilon_0$ represents the ratio between polar motion amplitudes for a liquid core-elastic mantle model and for a rigid earth.

Numerical values can be found in the following table, taken from Molodensky (1961) for Model 1.

	$\frac{\sigma + \Omega}{\Omega}$	β	$\varepsilon/\varepsilon_0$
Precession	0	198	1
Principal nutation	-1/6800	212	0.996
(18.7 years)	+1/6800	185	1.003
Semiannual nutation	-1/183	-125	1.088
	+1/183	57	1.036
Fortnightly nutation	-1/13.7	-4	1.080
	+1/13.7	+7	1.026

This table can be compared with the table on p. 122 of (Moritz, 1980b)

for the Poincaré model (rigid shell and liquid core); our present $\varepsilon/\varepsilon_0$ corresponds to U/U_0 there. The present values are much more in agreement with observation. The comparison shows that elasticity of the shell reduces the effect of the liquid core as compared to a rigid shell; this fact has already been pointed out by Jeffreys (1949).

Note the strong dependence of the (dimensionless) parameter β on frequency. Therefore, β is sometimes called Molodensky)s resonance parameter for the core.

The <u>proper frequencies</u> correspond to free oscillation without external force; for them we have $\kappa = 0$. Indeed, from $V_e = 0$ there follows $\kappa = 0$ by (8-16). As in the Poincaré model, there are four proper frequencies: the axial spin mode (ASM), σ_0 , the Chandler wobble (CW), σ_1 , the nearly diurnal free wobble (NDFW), σ_2 , and the turn-over mode (TOM), σ_3 ; cf. (Smith, 1977; Moritz, 1980b, p. 116). The "trivial" modes ASM and TOM are independent of the constitution of the body; they are also now the same as for a rigid body, namely

$$\sigma_0 = 0 , \qquad \sigma_3 = -\Omega \quad ; \tag{10-52}$$

cf. (Moritz, 1980b, p. 83).

The proper frequency for NDFW is obtained by putting $\kappa = 0$ in (10-42). Molodensky gives for his Model 1 the value

$$\sigma_2 = -1.00214\Omega \qquad \text{(Molodensky 1).} \tag{10-53}$$

This may be compared to the value

$$\sigma_2 = -\Omega\left(1 + \frac{e^2}{2}\frac{C}{C - C_c}\right) \tag{10-54}$$

for the Poincaré model, cf. (Moritz, 1980b, p. 116), with $\varepsilon = e^2/2$ and $C_m = C - C_c$. With the values (10-47) this gives

$$\sigma_2 = -1.00289\,\Omega \quad \text{(Poincaré)} . \tag{10-55}$$

For the Chandler frequency (CW) we cannot neglect $\sigma + \Omega$ so that we must use (10-41). In fact, the Chandler frequency is so small that we can neglect it as compared to Ω, so that now

$$\frac{\sigma + \Omega}{\Omega} = 1 + \frac{\sigma}{\Omega} \doteq 1 \quad . \tag{10-56}$$

The second equation of (10-46) may be written

$$\frac{1}{2}(\gamma - \beta)\, e^2 \frac{\sigma}{\Omega} + \frac{\Omega + \sigma}{\Omega}\beta - \varepsilon = 0 \quad . \tag{10-57}$$

For small σ/Ω, the first term may be neglected, and there remains

$$\beta \doteq \frac{\Omega}{\Omega + \sigma}\varepsilon \quad . \tag{10-58}$$

From (2-58) there follows

$$\frac{\sigma\nu}{2\Omega} = \frac{\sigma + \Omega}{\Omega}\beta - \varepsilon \doteq \varepsilon - \varepsilon = 0 \; . \tag{10-59}$$

With $\kappa = 0$ (proper frequency, no external potential) and (10-58) and (10-59), eq. (10-41) reduces to

$$\varepsilon - \frac{\sigma+\Omega}{\Omega}\left(\frac{A-A_c}{C}\varepsilon + \frac{A_c}{C}\frac{\Omega}{\Omega+\sigma}\varepsilon + \frac{C-A}{C}\frac{k}{k_s}\varepsilon\right) = 0 \ .$$

We divide by ε and solve for σ. The result is, writing $\sigma = \sigma_1$ for CW :

$$\frac{\sigma_1}{\Omega} = \frac{C-A}{A - A_c + \frac{k}{k_c}(C-A)}\left(1 - \frac{k}{k_s}\right) \ . \qquad (10\text{-}60)$$

To an error of about 1 % , this equals

$$\frac{\sigma_1}{\Omega} = \frac{C-A}{A-A_c}\left(1 - \frac{k}{k_s}\right) \ , \qquad (10\text{-}61)$$

a result given by Molodensky (1961). For his Model 1 he gets a value of σ_1 which corresponds to a Chandler period of 433 days.

Introducing the Euler frequency σ_E by

$$\sigma_E = \frac{C-A}{A}\Omega \qquad (10\text{-}62)$$

(Moritz, 1980b, p. 10), (10-60) may be written

$$\sigma_1 = \frac{1 - \frac{k}{k_s}}{1 - \frac{A_c}{A} + \frac{k}{k_s}\frac{\sigma_E}{\Omega}}\sigma_E \ . \qquad (10\text{-}63)$$

<u>Relation to McClure's theory.</u> This theory (McClure, 1973), also described in (Moritz, 1980b), is based on an entirely elastic model without liquid core. Such a model can be considered a limit of a Molodensky model for vanishing core:

$$b = 0 , \qquad A_c = C_c = 0 \quad . \tag{10-64}$$

Then (10-63) for the Chandler frequency reduces to

$$\sigma_1 = \frac{1 - \frac{k}{k_s}}{1 + \frac{k}{k_s}\frac{\sigma_E}{\Omega}}\,\sigma_E = \sigma_C \ , \tag{10-65}$$

identical to eq. (4-5) of (Moritz, 1980b).

Eq. (10-41) reduces to

$$\varepsilon - \frac{\sigma + \Omega}{\Omega}\left(\frac{A}{C}\varepsilon + \frac{C - A}{C}\frac{k}{k_s}\varepsilon\right) = \kappa\,\frac{C - A}{C\Omega^2}\left(1 - \frac{k}{k_s}\frac{\sigma + \Omega}{\Omega}\right) . \tag{10-66}$$

We multiply by

$$\frac{C}{A} = 1 + \frac{\sigma_E}{\Omega} \tag{10-67}$$

(by (10-62)) and introduce the tidal frequency ω_j and the nutational frequency $\Delta\omega_j$ by

$$\begin{aligned} \omega_j &= -\sigma \ , \\ \Delta\omega_j &= \omega_j - \Omega = -(\sigma + \Omega) \end{aligned} \tag{10-68}$$

(Moritz, 1980b, pp. 95 and 97). We get

$$\varepsilon\left[1+\frac{\sigma_E}{\Omega}+\frac{\Delta\omega_j}{\Omega}\left(1+\frac{k}{k_s}\frac{\sigma_E}{\Omega}\right)\right]=\kappa\frac{\sigma_E}{\Omega^3}\left(1+\frac{k}{k_s}\frac{\Delta\omega_j}{\Omega}\right) . \quad (10\text{-}69)$$

Using (10-65), this is readily seen to be the same as

$$\varepsilon=\frac{\sigma_E}{\Omega^2(\omega_j+\sigma_C)}\,\frac{1+\frac{k}{k_s}\frac{\Delta\omega_j}{\Omega}}{1+\frac{k}{k_s}\frac{\sigma_E}{\Omega}}\,\kappa\ . \quad (10\text{-}70)$$

This equation is equivalent to the tidal term in eq. (5-24) of (Moritz, 1980b, p. 34), as we shall see by relating the constant κ to the expansion of the tidal potential (ibid., sec. 1). For the tesseral part v_{21} of degree 2 and order 1 we have (ibid., eqs. (1-10) and (5-14)):

$$v_{21}=\frac{1}{3}\Omega^2a^2P_{21}(\cos\theta)\sum_j B_j\sin(\omega_jt+\beta_j+\lambda)\quad , \quad (10\text{-}71)$$

the sum being extended over all tidal frequencies ω_j, and B_j and β_j denoting amplitude and phase, respectively.

On the other hand, in the present report we have used the form (2-43) for the tidal potential:

$$\begin{aligned}V_e &= \kappa r^2\sin\theta\cos\theta\cos(\sigma t-\lambda)\\ &= \frac{1}{3}\kappa\, r^2P_{21}(\cos\theta)\cos(\sigma t-\lambda)\quad . \qquad (10\text{-}72)\end{aligned}$$

The comparison with (10-71) shows that (10-72) represents any term of the sum (10-71) with $\sigma = -\omega_j$ by (10-68) and

$$\kappa = \Omega^2 B_j \quad , \qquad \beta_j = 90^\circ \tag{10-73}$$

with $r = a = 1$ on the earth's surface.

Now, McClure's theory, eq. (5-24) of (Moritz, 1980b), limited to the tidally induced term with frequency ω_j , gives

$$m = i \frac{\sigma_E}{\omega_j + \sigma_C} \frac{1 + \frac{k}{k_s}\frac{\Delta\omega_j}{\Omega}}{1 + \frac{k}{k_s}\frac{\sigma_E}{\Omega}} B_j e^{-i(\omega_j t + \frac{\pi}{2})}$$

$$= \frac{\sigma_E}{\omega_j + \sigma_C} \frac{1 + \frac{k}{k_s}\frac{\Delta\omega_j}{\Omega}}{1 + \frac{k}{k_s}\frac{\sigma_E}{\Omega}} B_j e^{i\sigma t} \quad . \tag{10-74}$$

By (1-28) we have

$$m = m_1 + i m_2 = \varepsilon(\cos\sigma t + i\sin\sigma t) = \varepsilon e^{i\sigma t} \quad . \tag{10-75}$$

By (10-73) and (10-75) we thus see that (10-70) is indeed equivalent to (10-74), as was to be shown.

Relation to Poincaré's theory. Let us specialize eqs. (10-46) for the Poincaré model featuring a rigid mantle and a homogeneous liquid core. Since the rigid mantle is not deformed, we have

$$H(b) = 0 \quad , \tag{10-76}$$

and the first equation of (10-46) gives

$$\gamma = 0 \quad . \tag{10-77}$$

The second equation then yields

$$\frac{\sigma + \Omega}{\sigma} \beta - \frac{\Omega}{\sigma} \varepsilon = \frac{1}{2} e^2 \beta \ , \tag{10-78}$$

or

$$\beta + \frac{\Omega}{\sigma} (\beta - \varepsilon) = \frac{1}{2} e^2 \beta \ , \tag{10-79}$$

identical to (4-64). Since the rigid mantle is not deformed, the gravitational potential does not change, so that

$$V_1 = 0 \tag{10-80}$$

and therefore

$$k = 0 \quad . \tag{10-81}$$

Hence the third equation of (10-46) becomes

$$\varepsilon - \frac{\sigma + \Omega}{\Omega} \left[\frac{A - C_c}{C} \varepsilon + \left(\frac{C_c - A_c}{C} \frac{\sigma}{\Omega} + \frac{C_c}{C} \right) \beta \right] = \kappa \frac{C - A}{C\Omega^2} \ . \tag{10-82}$$

In sec. 4 we have shown that (10-79), or (4-64), is equivalent to

$$A_c \sigma \varepsilon + (A_c \sigma + C_c \Omega)(\beta - \varepsilon) = 0 \ ; \tag{10-83}$$

cf. (4-67). Using (4-61) and(4-69), namely

$$u_0 = \Omega\varepsilon \quad , \qquad v_0 = \Omega(\beta - \varepsilon) \ , \tag{10-84}$$

this has been brought to the form (4-54),

$$A_c \sigma u_0 + (A_c \sigma + C_c \Omega) v_0 = 0 \ , \tag{10-85}$$

which is one of the two basic equations of Poincaré's theory.

Let us now show that the second basic equation, namely (4-53), can be derived from the Euler-Liouville condition (10-82). We put $\kappa = 0$ for free oscillations. Then, by elementary manipulations,

we can bring (10-82) into the form

$$[A\sigma - (C - A)\Omega - (\sigma + \Omega)C_c] \frac{\varepsilon}{\sigma + \Omega} + \left[A_c + \frac{\sigma + \Omega}{\Omega}(C_c - A_c)\right]\beta = 0 \ . \tag{10-86}$$

Since $\sigma + \Omega$ is very small for tidal frequencies, we may write

$$(\sigma + \Omega)C_c \doteq (\sigma + \Omega)A_c$$

and neglect

$$\frac{\sigma + \Omega}{\Omega}(C_c - A_c)$$

as the product of two small quantities. Thus (10-86) becomes

$$[A\sigma - (C - A)\Omega - A_c(\sigma + \Omega)]\varepsilon + A_c(\sigma + \Omega)\beta = 0 \ . \tag{10-87}$$

This can be written in the form

$$[A\sigma - (C - A)\Omega]\varepsilon + A_c(\sigma + \Omega)(\beta - \varepsilon) = 0 \ . \tag{10-88}$$

With (10-84) this finally becomes

$$[A\sigma - (C - A)\Omega]u_0 + A_c(\sigma + \Omega)v_0 = 0 \ , \tag{10-89}$$

identical to (4-53). This completes the derivation of the fundamental equations of Poincaré's theory from Molodensky's theory, showing the equivalence of both (completely different!) theories for the Poincaré model.

11. Recent Developments

Molodensky's theory has been serving as a starting point for a number of subsequent investigations. It has been refined and made more general, but also simplified and made more elegant. Thus Molodensky was pioneering in this field, as he was in the geodetic boundary-value problem.

The work of Shen and Mansinha (1976). They provided a uniform treatment of mantle and core by using the equations of elasticity (cf. sec. 6) throughout the earth, regarding the liquid core (or liquid outer core) as a limiting case for an elastic core with

$$\mu = 0 \tag{11-1}$$

(see end of sec. 5). This requires the consideration of the ellipticity in the elastic equations, since the ellipticity is essential for core resonance as we have repeatedly seen. The motion of the liquid core is treated in terms of spheroidal and toroidal oscillations in the way outlined in sec. 3.

Shen and Mansinha's results agree well with Molodensky's. Naturally, these authors have been able to use recent better models for the earth's interior, especially the core; cf. also (Melchior, 1980a;Smith, 1980).

The method of Sasao, Okubo and Saito (1980). They gave a particularly elegant formulation of Molodensky's by generalizing the well-known Poincaré equations for a rigid mantle and a liquid core; cf. (Moritz, 1980b), eqs. (12-22) with (12-32):

$$A\dot{u} - i(C - A)\Omega u + A_c(\dot{v} + i\Omega v) = L \ ,$$
$$A_c\dot{u} + A_c\dot{v} + iC_c\Omega v = 0 \ . \qquad (11\text{-}2)$$

Here A and C are the principal moments of inertia of the whole earth, A_c and C_c those of the core, the complex number L combines the x_1 and x_2 components of the lunisolar torque:

$$L = L_1 + iL_2 \ , \qquad (11\text{-}3)$$

and the complex numbers u and v represent the rotation of the earth and that of the core, respectively (ibid., p. 118); see also sec. 4 of the present report.

(Sasao et al., 1980) succeeded in finding a surprisingly simple equivalent of eqs. (11-2) for Molodensky's problem. Using the same notations as above, it reads (ibid., eqs. (37) and (32)):

$$A\dot{u} - i(C - A)\Omega u + A_c(\dot{v} + i\Omega v) + \Omega(\dot{c} + i\Omega c) = L \ ,$$
$$A_c\dot{u} + A_c\dot{v} + iC_c\Omega v + \Omega\dot{c}_c = 0 \ . \qquad (11\text{-}4)$$

The new feature is the appearance of the complex numbers

$$c = c_{13} + ic_{23} \ ,$$
$$c_c = c^c_{13} + ic^c_{23} \ , \qquad (11\text{-}5)$$

combining the products of inertia c_{13} and c_{23} for the earth

(cf. Moritz, 1980b, pp. 16-17) and for the core, respectively. These quantities can be expressed in terms of the functions (8-25) characterizing the elastic behavior of the mantle, by means of the boundary conditions discussed in sec. 9.

In fact, the first equation of (11-4) is equivalent to the Euler-Liouville condition, (10-38) or the third equation of (10-46), whereas the second equation of (11-14) is a more elegant version of the second equation of (10-46); cf. the relation to Poincaré's theory discussed in sec. 10.

The theory of (Sasao et al., 1980), as well as that of (Shen and Mansinha, 1976), is applicable to more general earth models than those used by Molodensky; again, the numerical results are similar.

The work of Smith and Wahr. Smith (1977) used the normal modes (proper modes, proper frequencies) of the earth to study polar motion and nutation. The most extensive, detailed and accurate application of this theory was made by Wahr (1979, 1981a,b,c).

Wahr (1981a) used differential equations of motion for an elastic earth (6-33) in the form (6-37). For the liquid outer core the same equations hold with (11-1), so that a uniform treatment of the whole earth is possible. Also, the ellipticity is consistently taken into account.

The differential equations of motion have the form

$$\rho(\ddot{\underline{u}} + 2\underline{\omega} \times \dot{\underline{u}}) = \underline{L}\,\underline{u} \quad , \tag{11-6}$$

where $\underline{L}$ is a linear operator acting on the displacement vector $\underline{u}$ involving first and second derivatives with respect to $\underline{x}_i$. The coordinate system used is the uniformly rotating nutation frame; cf. secs. 1 and 6.

These equations are solved for $\underline{u}$ subject to the boundary conditions (cf. secs. 8 and 9) that the normal displacement $\underline{u} \cdot \underline{n}$ ($\underline{n}$ is the unit vector normal to the boundary surface), the stress vector $\underline{P}\,\underline{n}$ (5-13), the change of gravitational potential, V_1, and the quantity (9-24) must be continuous across any boundary. Furthermore, the displacement vector $\underline{u}$ as such must be continuous across any "welded boundary". Such a welded boundary separates elastic regions of different density, etc., such as crust and mantle; the core-mantle boundary is not a welded boundary since the core consists of a liquid whose movement is restricted only by continuity of the normal displacement (it can more freely along the boundary).

Eqs. (11-6) are transformed to spherical coordinates, with ellipsoidal corrections being taken into account. The solution $\underline{u}$ is consequently expressed in the form (3-1) :

$$\underline{u} = \sum_{n}\sum_{m}(\underline{S}_n^m + \underline{T}_n^m) \quad , \tag{11-7}$$

as a linear combination of spheroidal and toroidal modes (again, with small corrections for ellipticity).

For nutation, the terms with $m = 1$ are responsible. Wahr uses the truncated form

$$\underline{u} = \underline{T}_1^1 + \underline{S}_2^1 + \underline{T}_3^1 \tag{11-8}$$

for his purpose. This form is also employed by Shen and Mansinha (1976), whereas Molodensky's theory is equivalent to using

$$\underline{u} = \underline{T}_1^1 + \underline{S}_2^1 \quad ; \tag{11-9}$$

The toroidal term $\underline{T}_1^1$ may be interpreted as a small spatial rotation. In fact, (3-10) can be written as

$$\underline{T}_1^1 = \underline{\theta} \times \underline{x} \tag{11-10}$$

with

$$\underline{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ 0 \end{bmatrix} = \begin{bmatrix} Tr^{-1}\sin\sigma t \\ -Tr^{-1}\cos\sigma t \\ 0 \end{bmatrix} ; \tag{11-11}$$

this representation holds generally for any $\underline{T}_1^1$. Such a vector product is characteristic for an infinitesimal rotation; cf. (Moritz, 1980b, p. 78). If the vector $\underline{\theta}$ does not depend on the radius vector r , then we have a rigid rotation of the body as a whole. This is the case if

$$Tr^{-1} = \text{const.} \quad \text{or} \quad T(r) = cr \quad , \tag{11-12}$$

c being a constant. (Otherwise, the spheres r = const. undergo different rotations.)

For a homogeneous liquid core, T is indeed proportional to r , as (3-17) shows. A rigid mantle obviously also rotates as a whole, so that (11-12) holds. Of course, the rotations of core and mantle will in general be different, so that the constant c differs for core and mantle.

Remarkably enough, the relation (11-12) also holds, at least approximately, for far more general earth models featuring, e.g., an elastic mantle, a liquid outer core, and an elastic inner core. Each of these three regions is undergoing a nearly rigid rotation; cf. Fig. 7 in (Smith, 1977).

Using (11-10) we may write (11-8) in the form

$$\underline{u} = \underline{\theta} \times \underline{x} + \underline{u}' \quad . \tag{11-13}$$

The first term, $\underline{T}_1^1$, expresses a small rotation which will, by definition, be called <u>nutation</u> ; the remainder

$$\underline{u}' = \underline{S}_2^1 + \underline{T}_3^1 \tag{11-14}$$

will, again by definition, be called the effect of <u>body tide</u>. This splitting-up of the total displacement into nutation and tide is, to a certain degree, arbitrary but is in agreement with previous definitions of polar motion and nutation as we shall see below. A similar decomposition has been considered in sec. 3; cf. (3-13) and (3-21).

Thus nutation (rotation) and tidal deformation have been separated. Due to nutation, the earth's surface (again assumed spherical) rotates like a rigid sphere. Hence, as far as geometry (kinematics) is concerned, the rigid body theory of Smith (1977) and Wahr (1979, sec. 7), as extensively discussed in (Moritz, 1980b, secs. [illegible] and 11), may again be applied. In particular, the formulas for precession and polar motion (<u>ibid.</u>, eqs. (11-37) and (11-39)) can be used as far as they refer to purely kinematically defined

axes. This is true for the <u>rotation axis</u>, so that the first equation of (11-37), <u>ibid.</u>, remains valid:

$$n_R = i\frac{\sigma}{\Omega}\theta \quad , \tag{11-15}$$

where

$$\theta = \theta_1 + i\theta_2 \quad , \tag{11-16}$$

$$n_R = n_1 + in_2 \quad , \tag{11-17}$$

$$\underline{n}_R = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \underline{e}_R - \underline{e}_3 \quad . \tag{11-18}$$

Thus the vector $\underline{n}_R$ represents the very small difference between $\underline{e}_R$, the unit vector of the rotation axis, and $\underline{e}_3$, the coordinate unit vector of the z-axis of the reference frame (the nutation frame as mentioned above) which has a constant direction in space. The <u>complex number</u> n_R combines the first two components of the <u>spatial vector</u> $\underline{n}_R$ (the third component, n_3 , is easily seen to be negligibly small).

With the figure axis F we must be careful. From (Moritz, 1980b) we take the second equation of (11-37):

$$n_F = -i\theta \quad , \tag{11-19}$$

which is equivalent to

$$\underline{n}_F = \underline{e}_F - \underline{e}_3 = \underline{\theta} \times \underline{e}_3 \quad ; \qquad (11\text{-}20)$$

cf. ibid., eqs. (11-15) and (11-35). Thus $\underline{e}_F$ is obtained by rotating the nutation axis $\underline{e}_3$ by $\underline{\theta}$; cf. (11-10) with $\underline{e}_3$ taken for $\underline{x}$. In other terms, $\underline{e}_F$ is the unit vector of the z-axis to which the body tide $\underline{u}'$ refers. Assuming for the moment that $\underline{u}'$ refers to the body frame as defined at the end of sec. 6 and used by Molodensky, we can identify the axis F with the z-axis of the body frame; more about this will be said below. In particular, the axis F is an average axis of figure rather than the instantaneous axis of maximum inertia.

Polar motion is referred to the body axis $\underline{e}_F$, hence it is expressed by the complex number

$$p_R = n_R - n_F = i\,\frac{\sigma + \Omega}{\Omega}\,\theta \quad , \qquad (11\text{-}21)$$

in agreement with ibid., eq. (11-39).

The analogy with rigid-body motion breaks down for the angular momentum axis H , which is not a kinematical but a dynamical quantity. Hence, (11-37), ibid., does not hold for n_H . However, since the basic relation between the angular momentum vector $\underline{H}$ and the torque $\underline{L}$,

$$\frac{dH}{dt} = \underline{L} \quad , \qquad (11\text{-}22)$$

is independent of the physical constitution of the body, the relation (13-37), ibid., valid for a rigid body as well as for Poincaré's model,

$$n_H = -\frac{iL}{C\Omega(\sigma+\Omega)} , \tag{11-23}$$

must hold also in the present case. The corresponding polar motion is then given by

$$p_H = n_H - n_F \tag{11-24}$$

with (11-19) and (11-23).

This behavior of the kinematical axes R and F and of the dynamical axis H is fully analogous to the case of the Poincaré model; cf. (Moritz, 1980b, pp. 128 - 130).

Relation to Kinoshita's theory. Throughout the present report we have restricted our considerations to one individual frequency

$$\sigma = -\omega_j$$

only; the general case is then a sum of the individual frequency components; cf. also (Moritz, 1980b, pp. 95 - 98 and 140 - 141). Thus, e.g., θ has the usual exponential form

$$\theta = \eta e^{i(\sigma t+\gamma)}, \tag{11-25}$$

where the amplitude η is a real constant if (11-12) holds (or a function of r otherwise); for generality we have added a phase angle γ .

This complex quantity θ holds for the Molodensky-Wahr model.

For the rigid earth we have a similar quantity given by (ibid., eq. (11-33)) and denoted by θ_r ; its amplitude will be designated by η_r . Wahr (1979, 1981c) writes

$$\eta = \eta_r \left(1 + \frac{\eta - \eta_r}{\eta_r} \right) \tag{11-26}$$

and uses his theory only to calculate the small quantity $(\eta - \eta_r)/\eta_r$; η_r itself is obtained from the extremely accurate rigid-body theory of Kinoshita (1977); cf. also (Moritz, 1980b, sec. 9). The computational advantages of this procedure with regard to accuracy are evident.

Remark on reference systems. The body frames used in (Moritz, 1980b) and in the present report have always been Tisserand axes. For rigid body, such axes are fixed to that body, so that a rigid body is at rest with respect to Tisserand axes. For a nonrigid body, Tisserand axes are defined in such a way that the body is "on the average" at rest with respect to such a system.

A plausible mathematical formulation of "being at rest on the average" is obtained by condition

$$\iiint_v \rho |\underline{u}'|^2 dv = \text{minimum} \; , \tag{11-27}$$

minimizing the residual distortions $\underline{u}'$ in (11-13); ρ is the density and dv is the volume element. This condition can be shown to be basically identical to (ibid., eq. (3-7)).

The obvious choice of the volume of integration v is

$$v = \text{the whole earth} \quad . \tag{11-28a}$$

This choice is appropriate for a completely elastic earth without liquid core; the body axes xyz in secs. 3 to 8 of (Moritz, 1980b) have been defined in such a way.

In the case of a liquid core it is better to link the axes to the mantle only, by taking

$$v = \text{the mantle (plus crust)} \quad . \tag{11-28b}$$

The symmetry axes of the rigid mantle used as coordinate frame for Poincaré's theory (ibid., secs. 12 - 13) are such axes, and so is the body frame used in Molodensky's theory: the condition (10-13) is equivalent to (11-27) with (11-28b); cf. (ibid., p. 15).

A third choice is

$$v = \text{the crust} \tag{11-28c}$$

(which may practically be identified with the earth's surface). In view of plate motion, etcs., it may be argued which of the definitions (11-28b) or (11-28c) is geophysically more appropriate; for an excellent discussion see (Smith, 1981).

At any rate, for a spherically symmetric elastic mantle, both definitions (11-28b) and (11-28c) are equivalent provided the condition (11-12) holds since then the mantle (plus crust) rotates as a whole.

For Wahr's theory, the definition (11-28c) is most natural since we have seen that it amounts to interpreting $\underline{T}_1^1$ as a rigid rotation of the spherical earth's surface. If according to

(11-13) we split off the effect of this rotation, $\underline{\theta} \times \underline{x}$, from the displacement $\underline{u}$, the residual displacement $\underline{u}'$ will refer to a system which contains no such rotation, that is, no term $\underline{T}_1^1$, and this may be shown also to hold for Tisserand axes (another expression of the body's "being on the average at rest" in such a system). Thus $\underline{u}'$ refers to "crustal Tisserand axes" (11-28c), which practically coincide with Molodensky's "mantle Tisserand axes" since (11-12) holds to very high accuracy. Hence Wahr's $\underline{u}'$ is practically identical to Molodensky's $\underline{u}$, both referring to a body frame which is a crust-mantle Tisserand frame, whereas Wahr's original $\underline{u}$ refers to the nutation frame as we have pointed out above.

In the notation of eq. (11-20), the unit vector $\underline{e}_F$ has the direction of the z-axis of the body frame thus defined. As a Tisserand axis it is optimally stable with respect to the body, whereas the real instantaneous figure axis, the instantaneous axis of maximum inertia, is very unstable. This situation is completely analogous to the purely elastic case (Moritz, 1980b, pp. 40 and 49 - 50). As a matter of fact, the z-axis, represented by the vector $\underline{e}_F$, can be considered an <u>average</u> axis of figure. This is particularly obvious from Fig. 6.1. (<u>ibid.</u>, p. 38): the daily average of the instantaneous figure axis F is F_0 , and the average of F_0 over a Chandler period is 0 , corresponding to the z-axis.

For the sake of completeness we point out that also the instantaneous rotation axis R represents an average, although in a completely different sense. In fact, in a nonrigid body, each volume element undergoes a differential rotation described by the tensor $\underset{\sim}{R}$ given by (3-23) or (5-5), which in general varies

from point to point. Such a differential rotation is superimposed on the general body rotation and makes the rotational speed slightly different at each point.

Thus even the "instantaneous rotation veccor $\underline{\omega}$ of the earth as a whole" can only be an average over the whole earth of the rotation vectors of all volume elements. Jeffreys (1970, sec. 7.04) proposes to define $\underline{\omega}$ by the condition

$$\iiint \rho |\underline{V} - \underline{\omega} \times \underline{x}|^2 dv = \text{minimum} \quad , \tag{11-29}$$

where $\underline{V}$ is the velocity of any point with respect to an inertial system. This definition is intimately related to the definition of Tisserand axes, and our p_R (11-21) corresponds precisely to it.

Thus the rotation axis R represents a space average but is instantaneous in time, whereas the body axis z mentioned above is average of the figure axis over a long time period.

PART C

JEFFREYS' VARIATIONAL APPROACH

12. Variational Principles

The variational approach, initiated by Jeffreys (1949), has led to the poineering work (Jeffreys and Vicente, 1957a,b), which gave the first numerical results for nutation and tidal Love numbers for an earth model consisting of a liquid core and an elastic mantle. To my knowledge, this method has never been used again, partly because these papers are very difficult to read, as has been remarked by several authors, beginning with Molodensky (1961). Besides its historical importance, however, Jeffreys' method has considerable theoretical interest, and to the patient and detective-minded reader, the brilliant obscurity of his papers is a source of fascination (though sometimes mixed with frustration).

We shall here try to provide a simple introduction to Jeffreys' approach, pointing out the basic ideas and, perhaps, encouraging

and facilitating the study of the original papers. We begin with some well-known facts from theoretical mechanics, partly following (Courant and Hilbert, 1953, pp. 242-249).

Hamilton's principle. Consider a mechanical system of n degrees of freedom, described by n parameters (generalized or Lagrangian coordinates) $q_1, q_2, \ldots, q_n$. Then the motion of the system between two instants t_0 and t_1 of time t is such that

$$\int_{t_0}^{t_1} (T - U)dt = \text{extremum} \quad . \tag{12-1}$$

Here U denotes the potential energy, considered as a function of the coordinates and the time:

$$U = U(q_1, q_2, \ldots, q_n, t) = U(q_r, t) \quad , \tag{12-2}$$

and T denotes the kinetic energy, which is quadratic in the velocities $\dot{q}_r = dq_r/dt$:

$$T = \frac{1}{2} a_{rs}\dot{q}_r\dot{q}_s \quad , \tag{12-3}$$

where, in general, the coefficients a_{rs} are functions of the coordinates and the time :

$$a_{rs} = a_{rs}(q_1, q_2, \ldots, q_n, t) \quad . \tag{12-4}$$

In (12-3) and in the following we shall use the summation convention, summing over repeated indices (here, r and s) from 1 to n.

Introducing the Lagrangian function L by

$$L = T - U = L(q_r, \dot{q}_r, t) \quad , \tag{12-5}$$

we may write (12-1) as

$$\int_{t_0}^{t_1} L dt = \text{extremum} \tag{12-6}$$

or

$$\delta L = 0 \quad , \tag{12-7}$$

an extremum being equivalent to a vanishing variation δL as known from variational calculus. This condition leads to <u>Lagrange's equations</u> :

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r} = 0 \ . \tag{12-8}$$

An important special case is that of <u>small variations</u>. In this case, the a_{rs} in (12-3) are constants, and the potential energy (12-2) is a quadratic function of q_r. Thus

$$T = \frac{1}{2} a_{rs}\dot{q}_r\dot{q}_s \ ,$$
$$U = \frac{1}{2} b_{rs}q_rq_s \ , \qquad (12\text{-}9)$$

with constant coefficients a_{rs} and b_{rs} . With (12-9), Lagrange's equations (12-8) give

$$a_{rs}\ddot{q}_s + b_{rs}q_s = 0 \ , \qquad (12\text{-}10)$$

a system of n linear second-order ordinary differential equations with constant coefficients for the unknown coordinate functions $q_r = q_r(t)$.

<u>Vibrating string.</u> This case provides a continuous analogue to the discrete case of n degrees of freedom (Fig. 12.1). The

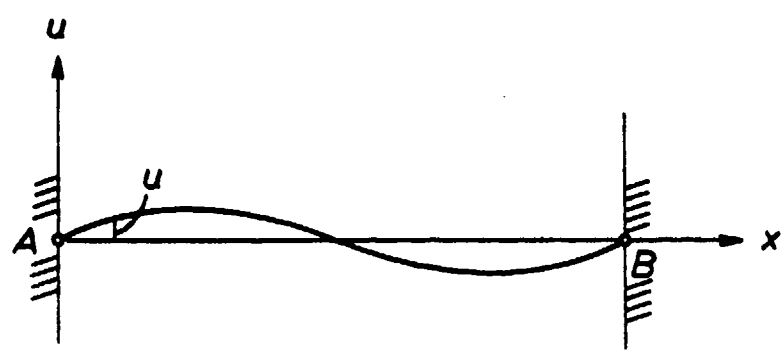

FIG. 12.1 Vibrating string.

string is kept fixed at the two end points A and B . Then the kinetic energy T and the potential energy U are given by the expressions

$$T = \frac{1}{2}\int_0^{\ell} \rho \dot{u}^2 dx \quad ,$$
$$U = \frac{1}{2}\int_0^{\ell} \mu u_x^2 dx \quad , \tag{12-11}$$

where

$$\dot{u} = u_t = \frac{\partial u}{\partial t} \; , \qquad u_x = \frac{\partial u}{\partial x} \quad , \tag{12-12}$$

ρ and μ are constants denoting density and tension of the spring, respectively, and $\ell = AB$.

The expressions (12-11) may be considered continuous analogues of (12-9), the integrals corresponding to the sums in (12-9) (implied by the summation convention).

Hamilton's principle

$$\int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} (T - U) dt = \text{extremum} \tag{12-13}$$

with (12-11), or $\delta L = 0$, can be shown by standard methods of variational calculus to lead to the linear partial differential equation

$$\rho u_{tt} - \mu u_{xx} = 0 \quad , \tag{12-14}$$

describing the motion of a vibrating string, which may be solved for the displacement

$$u = u(x,t) \quad . \tag{12-15}$$

Vibrating membrane. The step from the one-dimensional string to the two-dimensional membrane is evident: we have

$$T = \frac{1}{2} \iint_G \rho \dot{u}^2 dxdy \quad ,$$
$$U = \frac{1}{2} \iint_G \mu (u_x^2 + u_y^2) dxdy \quad , \tag{12-16}$$

G being the area of the membrane. Hamilton's principle $\delta L = \delta(T - U) = 0$ leads to the linear partial differential equation

$$\rho u_{tt} - \mu(u_{xx} + u_{yy}) = 0 \tag{12-17}$$

for the displacement

$$u = u(x,y,t) \quad . \tag{12-18}$$

Jeffreys' variational principle. Now we go to three-dimensions and a displacement vector $\underline{u} = (u_1, u_2, u_3)$. Then the kinetic energy is expressed by

$$T = \frac{1}{2} \iiint_V \rho |\dot{\underline{u}}|^2 dv = \frac{1}{2} \iiint_V \rho (\dot{u}_1^2 + \dot{u}_2^2 + \dot{u}_3^2) dv \quad , \tag{12-19}$$

where

$$dv = dxdydz = dx_1dx_2dx_3 \tag{12-20}$$

is the volume element and v the volume of integration (in our case, the earth). The potential energy is given by the expression

$$U = \frac{1}{2}\iiint_v \rho \left(\frac{\partial^2 V}{\partial x_i \partial x_j} u_i u_j + \frac{\partial(V_e + V_1)}{\partial x_i} u_i - \frac{\partial V}{\partial x_i} u_i \frac{\partial u_j}{\partial x_j} + \right.$$

$$\left. + \frac{\partial V}{\partial x_j} u_i \frac{\partial u_j}{\partial x_i} - \rho^{-1} p_{ij} \frac{\partial u_i}{\partial x_j} \right) dv \quad , \tag{12-21}$$

where V denotes the gravitational potential and ρ the density as used in sec. 6; the stress tensor p_{ij} is expressed in terms of u_i in the usual way (6-36).

The expression (12-19) for the kinetic energy is immediately obvious, being a natural generalization of the first equation of (12-16). The quantity U is quadratic in $\underline{u}$ and its derivatives, similarly to the second equation of (12-16), but is otherwise much more complicated. Jeffreys and Vicente (1957a) derived it from energy considerations. We shall here be satisfied with the fact that it provides the correct differential equations.

In fact, the condition $\delta L = 0$ leads by standard variational methods to

$$\rho \ddot{u}_i = -\rho \frac{\partial u_k}{\partial x_k} \frac{\partial V}{\partial x_i} + \rho \frac{\partial u_k}{\partial x_i} \frac{\partial V}{\partial x_k} + \rho u_k \frac{\partial^2 V}{\partial x_i \partial x_k} +$$

$$+ \rho \frac{\partial}{\partial x_i} (V_e + V_1) + \frac{\partial p_{ik}}{\partial x_k} \quad , \qquad (12\text{-}22)$$

which is the system of equations of motion for an elastic non-rotating earth, identical to (6-33) together with (6-37), for $\underline{\omega} = 0$ and $W = V$, corresponding to the absence of rotation and centrifugal force. This system is to be solved for the displacement vector

$$u_i = u_i(x_1, x_2, x_3, t) \quad . \qquad (12\text{-}23)$$

In this way the earth is considered a three-dimensional vibrating body whose oscillations are excited by the lunisolar force (the free oscillations of the earth are also described by these equations, with $V_e = 0$). The analogy to the vibrating membrane is obvious.

Finally we point out an important feature of the variational method : the condition $\delta L = 0$ gives not only a <u>differential equation</u> such as (12-22), but also <u>boundary conditions</u> (continuity of the normal displacement, of the stress vector, etc.). These boundary conditions are here obtained in a formal mathematical way, without recourse to physical considerations such as used in secs. 8 and 9 .

13. Method of Jeffreys and Vicente

Jeffreys (1949) and (Jeffreys and Vicente, 1957a, b) used the variational approach in an original and interesting way. They did not use the principle $\delta L = 0$ to derive (12-22) from (12-19) and (12-21), although they mention this possibility, but they try to reduce the continuous to the discrete case at the very beginning. We shall here try to outline the basic principles, referring the reader to the original paper for further details.

The idea. Starting again from the principle

$$\int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} (T - U) dt = \text{extremum} \quad , \qquad (13\text{-}1)$$

we try to reduce L from the continuous case, for which (12-19) and (12-21) hold, to the discrete case (12-9). This is done in the following way.

Assume that the elastic displacement vector $\underline{u}$ can be represented as a linear function of a finite number n of variables (Lagrangian parameters) $q_1, q_2, \ldots, q_n$. This is possible, e.g., by using the representation (11-7) in a truncated form such as (11-8). We thus write

$$u_i = \sum_{r=1}^{n} c_{ir} q_r = c_{ir} q_r \quad , \qquad (13\text{-}2)$$

using the summation convention; indices such as i or j will run from 1 to 3, corresponding to the coordinates in three-dimensional space, whereas r or s run from 1 to n. Let us make explicit the functional dependence:

$$\begin{aligned} u_i &= u_i(x_1, x_2, x_3, t) \quad , \\ c_{ir} &= c_{ir}(x_1, x_2, x_3) \quad , \qquad (13\text{-}3) \\ q_r &= q_r(t) \quad . \end{aligned}$$

The coefficients c_{ir} are considered known, the parameters q_r unknown.

Let us illustrate these general relations by the concrete case (11-8). The λ-component of $\underline{T}_1^1$, by (3-5) with (11-12), has the form

$$\begin{aligned} (\underline{T}_1^1)_\lambda &= -\, T(r)\cos\theta\sin(\sigma t - \lambda) \\ &= -r\cos\theta\cos\lambda \cdot c\,\sin\sigma t \\ &\quad + r\cos\theta\sin\lambda \cdot c\,\cos\sigma t \quad . \qquad (13\text{-}4) \end{aligned}$$

The first term on the right-hand side then contributes to the sum (13-2) with

$$\begin{aligned} c_{ir} &= c_{ir}(x_1, x_2, x_3) = r\cos\theta\sin\lambda \quad , \\ q_r &= q_r(t) = -\, c\,\sin\sigma t \quad . \end{aligned} \qquad (13\text{-}5)$$

Thus q_r represents essentially the unknown coefficient c. (It is clear that spherical coordinates r, θ, λ can be expressed in terms of x_1, x_2, x_3, so that c_{ir} is indeed a function of the latter; also, there is hardly any danger of confusing the subscript r with the radius vector.) Similar contributions come from the second term of (13-4) and, in fact, from any component of $\underline{S}_n^m$ and $\underline{T}_n^m$ in (11-8).

Having thus clarified the nature of c_{ir} and q_r, we substitute (13-2) into (12-19) and (12-21). The differentiations $\partial/\partial x_i$, etc., are performed on the known functions c_{ir}, and then the integrations over the earth's volume v are carried out, which do not affect the functions $q_r(t)$ because they are independent of position x_i. The result is a Lagrangian $L = T - U$ of form (12-5) which is quadratic in q_r because the original Lagrangian was quadratic in u_i. Then Lagrangian's equations (12-8) give a system of ordinary linear differential equations for q_r, similar to (12-10).

Consideration of rotation. This barest sketch of the basic idea must now be made more precise and more concrete. First of all, we must consider a rotating frame of reference. In fact, the initial equations (12-19) and (12-21) refer to a nonrotating inertial frame. They are transformed to the uniformly rotating nutation frame which Jeffreys and Vicente are using. Furthermore we must admit small rotations of the earth with respect to this frame, described by the vector $\underline{\theta}$ as in (11-13) or, equivalently, by small angles l', n', m' as in (4-8). We thus get an expression of form

$$L = T - U = \iiint_v F\left(u_i, \dot{u}_i, \frac{\partial u_i}{\partial x_j}, \ldots, l', m', n'\right) dv , \qquad (13\text{-}6)$$

similar to the one obtained by subtracting (12-21) from (12-19) but, in addition, containing rotation. The displacements u_i refer to the body frame and are thus the same as u_i in secs. 7 to 10, as well as the same as u_i' in (11-13); Jeffreys and Vicente write u_i' for our u_i in (13-6).

Secondly, we split up the integration over the earth into integrations over the core and over the mantle:

$$\iiint_v = \iiint_{\text{core}} + \iiint_{\text{mantle}} . \tag{13-7}$$

<u>The mantle.</u> Eqs. (7-17) give for the earth's surface $r = a = 1$:

$$u_r(1) = hS = hV_e \quad , \tag{13-8}$$

$$u_\theta(1) = l\partial S/\partial\theta = l\partial V_e/\partial\theta \quad , \tag{13-9}$$

here we have used (8-10), (8-11), and (8-16), putting

$$V_e = V_e(a) = V_e(1) \quad . \tag{13-10}$$

Here V_e is taken in the form (10-72) which depends on the tidal frequency

$$\sigma = -\omega_j \quad . \tag{13-11}$$

The Love numbers also depend on frequency, so that (13-8) should

be written more explicitly

$$u_r(1,\omega_j) = h(\omega_j)V_e(\omega_j) \quad . \tag{13-12}$$

The total displacement $u_r(1)$ is obtained by summin, over all (infinitely many) frequencies:

$$u_r(1) = \sum_j h_j V_e(\omega_j) \tag{13-13}$$

with

$$h_j = h(\omega_j) \quad . \tag{13-14}$$

Eq. (10-71) thus immediately gives (with $a = 1$)

$$u_r(1) = \frac{1}{3}\Omega^2 P_{21}(\cos\theta)\sum_j h_j B_j \sin(\omega_j t + \beta_j + \lambda) \quad . \tag{13-15}$$

Using the Legendre surface harmonics

$$\begin{aligned} R_{11}(\theta,\lambda) &= P_{11}(\cos\theta)\cos\lambda = 3\sin\theta\cos\theta\cos\lambda \quad , \\ S_{11}(\theta,\lambda) &= P_{11}(\cos\theta)\sin\lambda = 3\sin\theta\cos\theta\sin\lambda \quad , \end{aligned} \tag{13-16}$$

this may be written

$$u_r(1) = R_{11}(\theta,\lambda)q_1(t) + S_{11}(\theta,\lambda)q_2(t) \quad , \tag{13-17}$$

where

$$q_1(t) = \frac{1}{3}\,\Omega^2 \sum_j h_j B_j \sin(\omega_j t + \beta_j) \quad ,$$

$$q_2(t) = \frac{1}{3}\,\Omega^2 \sum_j h_j B_j \cos(\omega_j t + \beta_j) \quad . \tag{13-18}$$

In the same way we treat (13-9), obtaining

$$u_\theta(1) = \frac{\partial R_{11}(\theta,\lambda)}{\partial\theta}\, q_3(t) + \frac{\partial S_{11}(\theta,\lambda)}{\partial\theta}\, q_4(t) \tag{13-19}$$

with

$$q_3(t) = \frac{1}{3}\,\Omega^2 \sum_j l_j B_j \sin(\omega_j t + \beta_j) \quad ;$$

$$q_4(t) = \frac{1}{3}\,\Omega^2 \sum_j l_j B_j \cos(\omega_j t + \beta_j) \quad . \tag{13-20}$$

Consider finally the gravitational potential V_1 induced by the tidal potential V_e. By (8-18), (8-19), and (13-10) we have

$$V_1(1) = kV_e \quad . \tag{13-21}$$

Again, this is valid for one frequency only, and the total potential V_1 is the sum of all such frequency contributions. In the usual way we get

$$V_1(1) = R_{11}(\theta,\lambda)q_5(t) + S_{11}(\theta,\lambda)q_6(t) \qquad (13\text{-}22)$$

with

$$\begin{aligned} q_5(t) &= \frac{1}{3}\,\Omega^2 \sum_j k_j B_j \sin(\omega_j t + \beta_j) \quad , \\ q_5(t) &= \frac{1}{3}\,\Omega^2 \sum_j k_j B_j \cos(\omega_j t + \beta_j) \quad . \end{aligned} \qquad (13\text{-}23)$$

The functions $q_1(t), \ldots, q_6(t)$ will be taken as Lagrangian parameters for the mantle. For the reader familiar with spectral analysis it is obvious that they are essentially the transforms of the Love numbers

$$h_j = h(\omega_j) \quad , \qquad k_j = k(\omega_j) \quad , \qquad l_j = l(\omega_j) \qquad (13\text{-}24)$$

from the frequency domain into the time domain.

Next we consider the displacement vector $\underline{u}$ for general $r \leq 1$. On expressing the Love numbers by (13-8), (13-9), and (13-21) for one particular frequency ω_j :

$$\begin{aligned} h &= u_r(1)/S \quad , \\ l &= u_\theta(1)/S_\theta \quad , \\ k &= V_1(1)/S \quad , \end{aligned} \qquad (13\text{-}25)$$

substituting into (8-25), and inserting the resulting functions $H(r)$ and $L(r)$ into (7-17), we find u_r, u_θ, u_λ as linear

functions of $u_r(1), u_\theta(1)$, and $V_1(1)$. If we perform this operation for all frequencies ω_j and sum the results, we get in view of (13-17), (13-19), and (13-22) expressions of the form

$$u_r(r,\theta,\lambda,t) = Q_s(r,\theta,\lambda)q_s(t) \quad , \tag{13-26}$$

summation over s from 1 to 6 being implied, and similarly for u_θ and u_λ.

Finally we obtain the rectangular components u_1, u_2, u_3 by the rotation (3-8), which gives

$$u_i(x_1,x_2,x_3,t) = U_{is}(x_1,x_2,x_3)q_s(t) \quad , \tag{13-27}$$

expressing the displacements as linear functions of the Lagrangian variables q_s; the functions U_{is} are known.

Our choice of q_s, based on $u_r(1), u_\theta(1)$, and $V_e(1)$, has been motivated by the wish to have a close relation to the Love numbers h, k, l so that the developments in secs. 7 and 8 could be used. Jeffreys and Vicente (1957a) use a slightly different choice, based on $u_r(1), u_r(b)$, and $V_e(1)$, $u_r(b)$ being the radial displacement at the core-mantle boundary.

The core. By (4-4) we have

$$u_i = \alpha_{ij}x_j \tag{13-28}$$

(we may write x_j instead of x_j^o since u_j is small), where α_{ij} are given by (4-13) in the nutation frame. As usual we put $n = 0 = n'$, obtaining u_i as linear functions of l, m, l', m'.

In the body frame which we are using now, the u_i even depend on l and m only (the last rotation (4-8) is then missing). Thus

$$u_i = A_i l + B_i m \quad , \tag{13-29}$$

the coefficients being known functions of position as usual (even linear in x_i) .

Lagrange's equations. We now substitute (13-27) for the mantle and (13-29) for the core into (13-6) and perform the integration. The result has the form

$$L = L(q_1, q_2, \ldots, q_{10}) \tag{13-30}$$

with the Lagrangian variables q_1 to q_6 as above and with

$$\begin{aligned} q_7(t) &= l' \quad , \\ q_8(t) &= m' \quad , \\ q_9(t) &= l \quad , \\ q_{10}(t) &= m \quad . \end{aligned} \tag{13-31}$$

L will be quadratic in q_r for the reasons outlined at the beginning of this section; more precisely it is found to have the form

$$L = \frac{1}{2}\, a_{rs}\dot{q}_r\dot{q}_s - \frac{1}{2}\, b_{rs}q_r q_s + c_{rs}q_r\dot{q}_s - d_r q_r \tag{13-32}$$

with constant coefficients $a_{rs},\dots,d_r$; now, of course, the summation indices r and s each run from 1 to 10 .

The Lagrangian equations (12-8) give immediately

$$a_{rs}\ddot{q}_s + (c_{sr} - c_{rs})\dot{q}_s + b_{rs}q_s + d_r = 0 \quad . \tag{13-33}$$

The further treatment is standard. It is best to use complex combinations (note sin and cos in expressions such as (13-18)!)[1)]

$$\begin{aligned} q_1 + iq_2 &= \lambda_1 \quad , \\ q_3 + iq_4 &= \lambda_2 \quad , \\ q_5 + iq_6 &= \lambda_3 \quad , \\ q_7 + iq_8 &= \lambda_4 \quad , \\ q_9 + iq_{10} &= \lambda_5 \quad , \end{aligned} \tag{13-34}$$

and seek the solution in the form

$$\lambda_k = \lambda_k^o e^{i\sigma t} \qquad (k = 1,2,\dots,5) \tag{13-35}$$

(in view of (13-11) this amounts to a transformation into the frequency domain; a phase factor β_k is without importance). Then

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1) The deeper reason for the possibility of complex combination is the rotational symmetry of the earth model used.

$$\dot{\lambda}_k = i\sigma\lambda_k \quad , \qquad \ddot{\lambda}_k = -\sigma^2\lambda_k \quad , \tag{13-36}$$

and (13-33) reduces to a system of linear algebraic equations.

By (13-18) we have for one frequency $\sigma = -\omega_j$ only :

$$\lambda_1 = \frac{i}{3}\Omega^2 h_j B_j e^{-i(\omega_j t + \beta_j)}, \tag{13-37}$$

so that, apart fom constants, λ_1^o essentially is h (we again write h instead of h_j or $h(\omega_j)$). Similarly, λ_2^o and λ_3^o essentially are the Love numbers l and k , respectively. Our present λ_4 and λ_5 are immediately seen to be identical to λ' and λ in (4-29) and (4-46), so that (4-42) and (4-58) show that λ_4^o and λ_5^o are essentially (apart from costant factors) identical to Molodensky's parameters ε and β .

The system of linear algebraic equations to which (13-33) reduces, thus has the form

$$\begin{aligned}
&A_{11}h + A_{12}k + A_{13}l + A_{14}\varepsilon + A_{15}\beta = B_1 \ , \\
&A_{21}h + A_{22}k + A_{23}l + A_{24}\varepsilon + A_{25}\beta = B_2 \ , \\
&\quad\vdots \\
&A_{51}h + A_{52}k + A_{53}l + A_{54}\varepsilon + A_{55}\beta = B_5 \ ,
\end{aligned} \tag{13-38}$$

It is comparable to the system of 5 equations obtained by eliminating γ from the system of 6 equations (10-43) and (10-46) for $h,k,l,\varepsilon,\beta,\gamma$ and should be equivalent to it as a whole (al-

though the individual equations are different).

Each of the equations of sec. 10 has been derived separately from physical considerations (boundary conditions and Euler-Liouville equation). The present system (13-38), however, has been obtained as a whole directly from the variational principle $\delta L = 0$ using (13-32), in a formal way without physical considerations. (This is reminiscent of the fact that, also in the continuous case, the condition $\delta L = 0$ gives not only the respective differential equation but also corresponding boundary conditions, see the remark at the end of sec. 12!)

This difference between Molodensky's method and the method of Jeffreys and Vicente implies both advantages and drawbacks. In Molodensky's method, each condition has a simple physical meaning, it is thus more transparent and can be modified and extended to more complex earth models (e.g., containing a liquid outer core and an elastic inner core). Jeffreys' approach has a conceptually simpler structure: once the Lagrangian (13-6) has been established, everything else follows in a logically straightforward manner. Nevertheless, the details are enormously complicated, and only an author of the physical insight and mathematical skill of Sir Harold Jeffreys could devise such an approach and lead it to a successful conclusion.

In view of these difficulties, Jeffreys and Vicente (1957a,b) consider only two simplified earth models: the central particle model, consisting of a homogeneous core and a central mass point that is to represent the solid inner core, and the Roche model using a continuous density distribution in the core according to Roche's law. For the same reason, beginning at an early stage, computations are performed numerically instead of analytically:

for instance, already the coefficients $a_{rs}, \ldots, d_r$ in the Lagrangian (13-32) are calculated numerically. Equations such as (13-38) have no physical interpretation. Thus this approach, though logically more elegant, is physically less transparent. Also, as Molodensky (1961) has remarked, it is difficult to get a clear idea of the approximations involved in the present method.

The approach of Jeffreys and Vicente has not been followed by later investigators and is out of fashion nowadays. Still, the variational approach to partial differential equations in general is quite fashionable in both pure and applied mathematics because it is theoretically powerful and computationally convenient. Thus it does not seem impossible that Jeffreys' approach will some day be revived in a different form.

14. Numerical Aspects

Let us finally give some numerical values showing the practical implication of the theories discussed in the present report. We shall use the following abbreviations:

JV1	(Jeffreys and Vicente, 1957a)
JV2	(Jeffreys and Vicente, 1957b)
M1	(Molodensky, 1961), Model 1
M2	(Molodensky, 1961), Model 2
SM	(Shen and Mansinha, 1976)
W	(Wahr, 1979, 1981 b,c)
K	(Kinoshita, 1977), rigid earth .

Nutation. Astronomers use the angles $\Delta\varepsilon$ (nutation in obliquity) and $\Delta\psi$ (nutation in longitude) : $\varepsilon \doteq 23.5^\circ$ is the obliquity of the ecliptic. They are related to n_R or n_F by eq. (11-55) of (Moritz, 1980b), with $\theta = \varepsilon$.

From (McCarthy et al., 1980), Table 3, we take the following values for the nutation of the rotation axis n_R :

Term	Astronomically Observed	Rigid Earth	JV1	JV2	M1	M2
18.6 year $\Delta\varepsilon$	9.2050"±0.0017"	9.2273	9.1972	9.2145	9.1951	9.1985
$\Delta\psi \sin\varepsilon$	6.8409"±0.0025"	6.8713	6.8327	6.8556	6.8312	6.8356
semiannual $\Delta\varepsilon$	0.578" ±0.004"	0.5500	0.5706	0.5377	0.5710	0.5686
$\Delta\psi \sin\varepsilon$	0.533" ±0.004"	0.5046	0.5212	0.4859	0.5216	0.5196
fortnightly $\Delta\varepsilon$	0.0912"±0.0015"	0.0886	0.0914	0.0912	0.0913	0.0908
$\Delta\psi \sin\varepsilon$	0.0859"±0.0010"	0.0813	0.0835	0.0835	0.0834	0.0831

Note that these rigid-earth values, serving as a reference, do not correspond to Kinoshita.

(Wahr, 1981c, p. 725) gives for the nutation of the mean figure axis, n_F (which is the z-axis of the body system, a Tisserand axis for mantle or crust) :

Term	K	M2	SM	W
18.6 year $\Delta\varepsilon$	9.2278"	9.2044	9.2012	9.2025
$\Delta\psi \sin\varepsilon$	6.8743"	6.8441	6.8400	6.8416
semiannual $\Delta\varepsilon$	0.5534"	0.5719	0.5745	0.5736
$\Delta\psi \sin\varepsilon$	0.5082"	0.5232	0.5253	0.5245
fortnightly $\Delta\varepsilon$	0.0949"	0.0972	0.0978	0.0977
$\Delta\psi \sin\varepsilon$	0.0881"	0.0899	0.0904	0.0905

We see that it is necessary to take liquid-core effects into account if we want to get agreement with observations. A rigid-earth theory is no longer adequate. On the other hand, various modern liquid-core theories give very similar results; still, the differences on the order of 0.002" are becoming observationally significant.

At its General Assembly in Montreal in August 1979, the International Astronomical Union (IAU) has made the transition from a rigid earth to a liquid-core model and adopted nutational constants based on the Molodensky 2 model. The International Union of Geodesy and Geophysics (IUGG), at its General Assembly in Canberra in December 1979, has asked the IAU to reconsider this decision. It now appears probable that the IAU will reverse its decision and adopt values based on a Wahr model.

Nearly diurnal free wobble. Let this frequency be denoted by σ_2 (cf. (10-53)); then the following values for σ_2/Ω have been obtained:

	$-\sigma_2/\Omega$
JV1	1.00224
JV2	1.00403
M1	1.00214
M2	1.00216
W	1.00218

Chandler period. Empirical determinations give a period of about

(435±3) sidereal days

(Dahlen, 1980). Corresponding values from earth models are (in sidereal days)

	direct	corrected for ocean
JV1	392	430
JV2	395	433
M1	401	433
M2	402	436
W	403	

The effect of the oceans on the Chandler period is considerable. Dahlen (1976) found that oceans act to increase the Chandler period by 28 sidereal days. See also (Smith, 1977).

Love numbers. They are frequency dependent. We shall give them for some main diurnal tidal frequencies. For the notation O_1, P_1, K_1 cf. (Melchior, 1978, pp. 27 and 50-51); they correspond to nutational periods of 13.7 days (fortnightly), 183 days (semiannual), and ∞ (precession), respectively.

Love number h

	JV1	JV2	M1	M2	W
O_1	0.584	0.603	0.617	0.614	0.603
P_1	0.555	0.568	0.594	0.593	0.581
K_1	0.492	0.551	0.527	0.535	0.520

Love number k

	JV1	JV2	M1	M2	W
O_1	0.242	0.261	0.305	0.300	0.298
P_1	0.231	0.264	0.294	0.290	0.287
K_1	0.206	0.244	0.260	0.261	0.256

Love number l

	JV1	JV2	M1	M2	W
O_1	0.082	0.078	0.090	0.081	0.084
P_1	0.082	0.084	0.091	0.082	0.085
K_1	0.086	0.082	0.093	0.084	0.087

These values have been taken from (Molodensky, 1961) for JV1, JV2, M1 and M2, and from (Wahr, 1979); cf. also (Melchior, 1978, pp. 155 and 158).

Concluding remarks. These numerical values are shown only for the purpose of illustrating numerical differences between different theories. Since the various theories should be essentially equivalent, the discrepancies will be due primarily to different models for the earth's interior and to computational inaccuracies.

The earth models used range from the highly simplified models employed by Jeffreys and Vicente to modern sophisticated earth models used by Wahr. Therefore, and because of the precision of theory and computations, Wahr's results should be most reliable.

A discussion of the physical aspects of earth models used,

important as the subject is, is beyond the scope of the present report, and the reader is referred to the literature, e.g. (Smith, 1977; Melchior, 1980a,b; Wahr, 1981b).

In all these models, the earth is considered a sphere or an ellipsoid of revolution, and the mass distribution is to have a corresponding symmetry. Deviations of the actual earth from such an idealization can cause considerable discrepancies. For example, we have already mentioned the influence of the oceans on the Chandler period; ocean loading and other local effects also have a considerable influence on earth tides. The basic references for a geophysical discussion of the earth's rotation are (Munk and Macdonald, 1960) and (Lambeck, 1980).

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